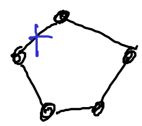
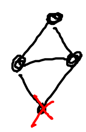
Critical graphs

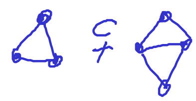
G is **k-critical** if $\chi(G) = k$ but
$(\forall G' \subsetneq G)\ \chi(G') \le k-1.$

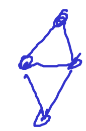
Example:  $\chi(C_5) = 3$

$(\forall G' \subsetneq C_5)\ \chi(G') \le 2$

C_5 is 3-critical.

 $\chi = 3$

 $\chi(\triangle) = 3$

 is not critical

Observation 1: $\chi(G) < k$ iff no subgraph of G is k-critical.

Proof: "$\Rightarrow$" easy

"$\Leftarrow$": If $\chi(G) \ge k$ $\quad H \subseteq G$ s.t. $\chi(H) \ge k$

H smallest possible

- every proper sg. of H is $(k-1)$-colorable
- $\chi(H) = k$

$\rightarrow$ H is k-critical.

2-critical ?

3-critical: $\chi(H) \ge 3 \Longrightarrow H$ contains an odd cycle

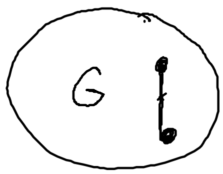
$\boxed{H \text{ 3-critical} \Longleftrightarrow H = \text{odd cycle}}$

Lemma 2: G k-critical $\Rightarrow \delta(G) \ge k-1$

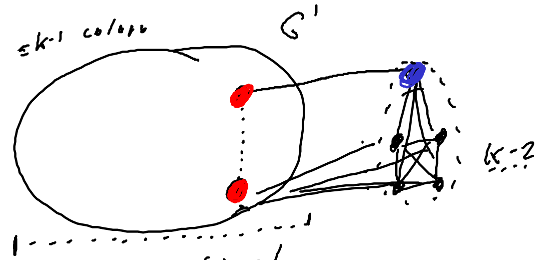
Proof: Suppose $v \in V(G)$ has degree $< k-1$

$\chi(G-v) \le k-1$, color v greedily $\rightarrow \chi(G) \le k-1$ ↯

K_k: k-critical, $(k-1)$-regular.



k-critical



k-critical

$\chi(G') \geq k$

$$e' = e - 1 + \frac{(k-2)(k-1)}{2} + k - 1 = e + \frac{k(k-1)}{2} - 1$$

$$n' = n + (k-1)$$

$$e_m = \binom{k}{2} + m \cdot \left(\frac{k(k-1)}{2} - 1\right)$$

$$n_m = k + m \cdot (k-1)$$

$$\text{avg degree} = \frac{2e_m}{n_m} = \frac{2\left(\binom{k}{2} + m\left(\frac{k(k-1)}{2} - 1\right)\right)}{k + m(k-1)} \xrightarrow[m \to \infty]{} \frac{2\left(\frac{k(k-1)}{2} - 1\right)}{k-1} = \boxed{k - \frac{2}{k-1}}$$

Theorem [Kostochka & Bernshteyn]

G k-critical $\Rightarrow$ avg degree of $G \geq k - \frac{2}{k-1} - O\left(\frac{1}{|V(G)|}\right)$

Theorem 3: G k-critical, $G \neq K_k$

$\Rightarrow$ average degree of $G \geq k - 1 + \frac{k-3}{k^2-3}$

Proof: Next time.

Critical graphs & coloring graphs on surfaces.

Generalized Euler's formula: If G drawn on surface of Euler genus g, s faces, then $|E(G)| \leq |V(G)| + s + g - 2$

$|V(G)| \geq 3$

Corollary If G is drawn on a surface of Euler genus g, then

$$\text{avg degree of } G \text{ is } \leq 6 + \frac{6 \cdot (g-2)}{|V(G)|}.$$

Proof: WLOG G is maximal $\Rightarrow$ each face has length ≥ 3

$$2|E(G)| = \sum_f |f| \geq 3 \cdot s \qquad s \leq \frac{2}{3}|E(G)|$$

$$|E(G)| \leq |V(G)| + s + g - 2 \leq |V(G)| + \frac{2}{3}|E(G)| + g - 2$$

$$\frac{1}{3}|E(G)| \leq |V(G)| + g - 2$$

$$\text{avg.deg of } G = \frac{2|E(G)|}{|V(G)|} \leq 6 + \frac{6(g-2)}{|V(G)|}$$

compare with Thm 3:

If G is 7-critical graph, $G \neq K_7$ then avg.deg. of $G \geq 6 + \frac{1}{11}$

Corollary: If G is 7-critical graph drawn on a surface of Euler genus g, then

$$|V(G)| \leq 66(g-2) \quad \text{or} \quad G = K_7$$

$$\cancel{6} + \frac{1}{11} \leq \text{avg deg} \leq \cancel{6} + \frac{6(g-2)}{|V(G)|}$$

Torus: → K_7 is the only 7-critical graph on torus

→ G on torus is 6-colorable iff $K_7 \not\subseteq G$.

In general: for fixed g — $\mathcal{F}_g = \{ F : F$ 7-critical, F drawn on a surface of Euler genus $g \}$ is finite

Decide if G is 6-colorable by checking whether G contains a subgraph belonging to $\mathcal{F}_g$

⇝ $O(|V(G)|)$ using clever subgraph testing.

G △-free, drawn on surf of Euler genus g, $|V(G)| \geq 3$, then avg. deg of $G \leq 4 + \frac{4(g-2)}{|V(G)|}$.

# of colors	2	3	4	5	6	7	
graphs on fixed surface	P	NP-complete →	?	P (Thomassen)	P	P	....
△-free on fixed surface	P	P (D., Král', Thomas)	P	P	P	P	
girth ≥ 5 on fixed surf	P	P (Thomassen)	P	P	P	P	
girth ≥ 6 on fixed surf.	P	P	P	P	P	P	

Grötzsch: Planar, triangle-free ⇒ 3-colorable

girth G ... length of the shortest cycle in G