Coloring graphs on surfaces by fixed number of colors

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1 Basic results

Theorem 1 (Four Color Theorem)**.** *Every planar graph is 4-colorable.*

Theorem 2. *Deciding whether a planar graph is* 3*-colorable is NP-hard.*

Theorem 3 (Grötzsch' Theorem)**.** *Every planar triangle-free graph is 3-colorable.*

Euler genus of a surface Σ is $2\times$number of handles$+$number of crosscaps needed to create Σ from the sphere. The embedding of a graph in Σ is 2-cell if all faces are homeomorphic to open disks.

Lemma 4 (Generalized Euler's Formula)**.** *A (multi)graph G embedded in a surface of Euler genus g with s faces satisfies*

$$\|G\| \le |G| + s + g - 2,$$

with equality iff the embedding is 2*-cell.*

The *girth* of a graph is the length of its shortest cycle.

Corollary 5. *A (multi)graph G embedded in a surface of Euler genus g such that each face has length at least $k \ge 3$ satisfies*

$$\|G\| \le \frac{k}{k-2}(|G| + g - 2).$$

Consequently, if G has girth at least k and G is not a forest, then

$$\|G\| \le \frac{k}{k-2}(|G| + g - 2);$$

and in particular, if G is simple (has no loops or parallel edges) and $|G| \ge 3$, then

$$\|G\| \le 3(|G| + g - 2).$$

Theorem 6 (Heawood's formula)**.** *A graph embedded in a surface Σ of Euler genus g has chromatic number at most*

$$\frac{7+\sqrt{24g+1}}{2}.$$

This bound is tight with the exception of Σ being the Klein bottle, in which case it can be improved to 6.

2 Coloring with fewer colors

By Corollary 5:

Theorem 7. *Let $C_{k,g} = \frac{2k(g-2)}{k-2}$. Let G be a graph with at least three vertices. If G has girth at least $k \geq 3$ and can be embedded in a surface of Euler genus g, then the average degree of G is at most*

$$\frac{2k}{k-2} + C_{k,g}/|G|;$$

and in particular, the average degree is at most

$$6 + C_{3,g}/|G|.$$

Corollary 8. *For every g and k, there exists $n_{g,k}$ as follows. If a graph of girth at least $k \geq 3$ embedded in a surface of Euler genus g has at least $n_{g,k}$ vertices, then G contains a vertex of degree at most $\lfloor 2k/(k-2) \rfloor \leq 6$.*

Corollary 9. *For every g and k, and any $c \geq \lfloor 2k/(k-2) \rfloor + 1$, there exists a linear-time algorithm to decide whether a graph of girth at least $k \geq 3$ embeddable in a surface of Euler genus g is c-colorable.*

Proof. If G contains a vertex v of degree less than c, then G is c-colorable if and only if $G - v$ is c-colorable. Otherwise, by Corollary 8 we have $|G| < n_{g,k}$, and we can test c-colorability of G by brute force in constant time. □

Hence, we have the following for graphs embedded in a general surface:

girth \ colors	3	4	5	6	≥7
3 (general)	NP-hard	?	?	?	P
4 (triangle-free)	?	?	P	P	P
5	?	P	P	P	P
≥6	?	P	P	P	P

To fill in a few more blanks, we need to consider coloring of graphs of average degree close to $2k/(k-2)$. It is more convenient to work in the setting of critical graphs.

3 Critical graphs

A graph G is *c-critical* if $\chi(G) = c$, but every proper subgraph of G is $(c-1)$-colorable.

Observation 10. *A graph is c-colorable if and only of it does not contain a* $(c+1)$*-critical subgraph.*

Observation 11. *Each* $(c+1)$*-critical graph has minimum degree at least* c*.*

By Corollary 8:

Corollary 12. *For every* g *and* k*, there exists* $n_{g,k}$ *such that if* G *is a* $(c+1)$*-critical graph of girth at least* $k \geq 3$ *for* $c \geq \lfloor 2k/(k-2) \rfloor + 1$*, and* G *can be embedded in a surface of Euler genus* g*, then* $|G| \leq n_{g,k}$*. In particular, for every* $c \geq \lfloor 2k/(k-2) \rfloor + 1$*, there are only finitely many* $(c+1)$*-critical graphs of girth at least* k *that can be embedded in a surface of Euler genus* g*.*

Combining Corollary 12 with Observation 10, we can give another proof to Corollary 9: it suffices to test whether G contains as a subgraph one of the finitely many $(c+1)$-critical graphs of girth at least k embeddable in the surface.

Observation 11 implies that a $(c+1)$-critical graph has average degree at least c. This bound is tight for the complete graph K_{c+1}, but this is the only example. In future lectures, we are going to prove the following improved bound.

Theorem 13. *If* G *is* $(c+1)$*-critical and* $G \neq K_{c+1}$*, then* G *has average degree at least*

$$c + \frac{c-2}{c^2+2c-2}.$$

Let us remark that even this bound can be improved by forbidding further small special graphs; the correct bound is $c+1-2/c+o(1)$, and we will discuss this in more detail later. It is easy to see (homework) that Theorem 13 implies that the number of 7-critical graphs, the number of triangle-free 5-critical graphs, and the number of 4-critical graphs of girth at least 6 that can be embedded in any fixed surface is finite. Consequently, for graphs embedded in a surface, the 6-colorability, the 4-colorability when the graph is triangle-free, and the 3-colorability when G has girth at least 6 is polynomial-time solvable.