Density of 4-critical graphs

Zdeněk Dvořák (based on a paper of A. Kostochka and M. Yancey)

March 24, 2018

For a graph G, let us define

$$p(G) := 5|G| - 3\|G\|.$$

Note that $p(K_1) = 5$, $p(K_2) = 7$, and $p(K_3) = 6$. Furthermore, if G' is a spanning subgraph of G, then $p(G') \geq p(G)$. Hence, $p(G) \geq 7$ for any graph with at most 3 vertices other than K_1 and K_3.

Theorem 1. *If G is 4-critical, then $p(G) \leq 2$, and thus*

$$\|G\| \geq \frac{5|G| - 2}{3}.$$

We say a graph G is a *counterexample* if G is 4-critical and $p(G) \geq 3$. The graph G is a *minimal counterexample* if G is a counterexample and there is no counterexample G' satisfying either $|G'| < |G|$, or $|G'| = |G|$ and $\|G'\| < \|G\|$. Since $p(K_4) = 2$, a minimal counterexample must have at least 5 vertices.

Lemma 2. *Let G be a minimal counterexample and let $S \neq V(G)$ be a set of its vertices of size at least 4. Let $S_0 \subseteq S$ be the set of vertices of S that have a neighbor in $V(G) \setminus S$. Then there exists a graph $G' \subseteq G$ such that $G[S] \subsetneq G'$ and $p(G') \leq p(G[S]) - 3$. Furthermore, if there exists a 3-coloring of $G[S]$ that uses at least two distinct colors on S_0, then either $p(G') \leq p(G[S]) - 4$ or $G' \neq G$.*

Proof. Let φ be a proper 3-coloring of $G[S]$ (which exists since G is 4-critical). If possible, choose φ so that at least two distinct colors appear on S_0.

Let G_1 be the graph obtained from G by adding a triangle $T = x_1x_2x_3$ and identifying all vertices in $\varphi^{-1}(i)$ with x_i for $i = 1, 2, 3$. Clearly, any 3-coloring of G_1 would give a 3-coloring of G, and since $\chi(G) = 4$, no such 3-coloring exists. Hence, G_1 has a 4-critical subgraph G_2. By the minimality of

G, the graph G_2 is not a counterexample, and thus $p(G_2) \leq 2$. Furthermore, G_2 is not a subgraph of G, and thus $T' := T \cap G_2$ is non-empty. Let G' be the graph obtained from G_2 by replacing T' with $G[S]$—we have $V(G') = (V(G_2) \setminus V(T')) \cup S$ and $E(G') = (E(G_2) \setminus E(T')) \cup E(G[S])$. Since $p(G_2) \leq 2$ and T' is a non-empty graph on at most three vertices, we have

$$p(G') = p(G_2) - p(T') + p(G[S]) \leq p(G[S]) - 3,$$

as required.

Furthermore, the equality holds only if $T' = K_1$; say $V(T') = x_1$. If additionally $G' = G$, then G_2 contains all edges between S_0 and $V(G) \setminus S$, and thus all these edges are incident with x_1. Equivalently, all vertices of S_0 have color 1 in the coloring φ. □

Lemma 3. *Let G be a minimal counterexample. If H is a proper subgraph of G with at least two vertices, then $p(H) \geq 6$.*

Proof. The claim is easy to verify when $|V(H)| \leq 3$. Let H be a proper subgraph of G with at least 4 vertices such that $p(H)$ is minimum; we only need to show that $p(H) \geq 6$.

If H is not an induced subgraph, then there exists an edge $e \in E(G) \setminus E(H)$ with both ends in $V(H)$. We have $p(H + e) < p(H)$; the minimality of $p(H)$ implies that $H + e = G$. Since G is a counterexample, we have $p(G) \geq 3$, and thus $p(H) = p(G) + 3 \geq 6$. Hence, we can assume that H is an induced subgraph of G.

Let $S = V(H)$. Since H is a proper (induced) subgraph, we have $S \neq V(G)$. By Lemma 2, there exists $G' \subseteq G$ such that $H = G[S] \subsetneq G'$ a $p(G') \leq p(H) - 3$. By the minimality of $p(H)$, we have $G' = G$, and thus $p(H) \geq p(G') + 3 = p(G) + 3 \geq 6$. □

Corollary 4. *Let G be a minimal counterexample. If H is a proper subgraph of G such that either $|H| < |G|$ or $\|H\| \leq \|G\| - 2$, then for any distinct $u, v \in V(H)$, the graph $H + uv$ is 3-colorable.*

Proof. If $H + uv$ is not 3-colorable, then it contains a 4-critical subgraph H', with either $|H'| < |G|$ or $\|H'\| \leq \|G\| - 1$. By the minimality of G, we conclude that H' is not a counterexample, and thus $p(H') \leq 2$, and $p(H' - uv) \leq 5$. Since H' is a proper subgraph of G with at least two vertices, this contradicts Lemma 3. □

Lemma 5. *Let G be a minimal counterexample, and let H be a proper subgraph of G. If $H \neq K_1, K_3$ and H is not obtained from G by removing one edge, then $p(H) \geq 7$.*

Proof. The claim is easy to verify when $|V(H)| \leq 3$. Let H be a proper subgraph of G with at least 4 vertices, not obtained from G by removing one edge, and such that $p(H)$ is minimum; we only need to show that $p(H) \geq 7$.

If H is not an induced subgraph, then there exists an edge $e' \in E(G) \setminus E(H)$ with both ends in $V(H)$. Since $p(H+e') < p(H)$, the minimality of $p(H)$ implies that $H + e' = G - e$ for some edge $e \in E(G)$. However, $p(G) \geq 3$, and thus $p(H) = p(G) + 6 \geq 9$. Hence, we can assume that H is an induced subgraph of G.

Let $S = V(H)$ and let $S_0 \subseteq S$ consist of the vertices with a neighbor in $V(G) \setminus S$. Since G is 4-critical, it is 2-connected, and thus $|S_0| \geq 2$. Let $u, v \in S_0$ be distinct, and let e_u and e_v be edges joining them to their neighbors in $V(G) \setminus S$. By Corollary 4, there exists a proper 3-coloring of $H + uv$; this coloring uses at least two distinct colors on S_0. By Lemma 2, there exists $G' \subseteq G$ such that either $p(G') \leq p(H) - 4$, or $p(G') = p(G) - 3$ and $G' \neq G$.

By the minimality of $p(H)$, we have either $G' = G - e$ for some edge $e \in E(G)$, or $G' = G$. If $G' = G - e$, then $p(H) \geq p(G') + 3 = p(G) + 6 \geq 9$. If $G' = G$, then $p(H) \geq p(G') + 4 = p(G) + 4 \geq 7$. □

Lemma 6. *If G is a minimal counterexample, then each triangle in G contains at most one vertex of degree* 3.

Proof. Suppose for a contradiction that $T = v_1v_2v_3$ is a triangle in G such that $\deg(v_1) = \deg(v_2) = 3$. Let x_1 and x_2 be the neighbors of v_1 and v_2 outside of T. If $x_1 = x_2$, then adding the edge x_1v_3 would create K_4, contradicting Corollary 4. Hence, $x_1 \neq x_2$. Let $G_1 = G - \{v_1, v_2\} + x_1x_2$. By Corollary 4, there exists a 3-coloring φ of G_1. Since $\varphi(x_1) \neq \varphi(x_2)$, we can by symmetry assume that $\varphi(x_1) = 1$, $\varphi(v_3) = 3$, and $\varphi(x_2) \in \{2, 3\}$. Coloring v_1 by 2 and v_2 by 1, we obtain a 3-coloring of G, which is a contradiction. □

Lemma 7. *If G is a minimal counterexample, $uv \in E(G)$, and* $\deg(u) = \deg(v) = 3$*, then u is contained in a triangle.*

Proof. Let x_1 and x_2 be the neighbors of u distinct from v. Suppose for a contradiction that $x_1x_2 \notin E(G)$. Let G_1 be the graph obtained from $G - \{u, v\}$ by identifying x_1 and x_2 to a single vertex x. Any 3-coloring of G_1 clearly extends to a 3-coloring of G; we conclude that G_1 is not 3-colorable, and thus it contains a 4-critical subgraph G_2. Since $G_2 \not\subseteq G$, we conclude that $x \in V(G_2)$. The minimality of G implies that G_2 is not a counterexample, and thus $p(G_2) \leq 2$. Let G_3 be the subgraph of G obtained from G_2 by decontracting x and adding the path x_1ux_2. We have $p(G_3) = p(G_2) + 4 \leq 6$. Furthermore, $v \notin V(G_3)$, and thus G_3 is a proper subgraph of G and it is not obtained from G by removing an edge. This contradicts Lemma 5. □

Corollary 8. *If G is a minimal counterexample, then each vertex of degree 3 has at most one neighbor of degree three.*

Proof. Suppose u is a vertex of degree 3 with neighbors v_1, v_2, and v_3. If $\deg(v_1) = 3$, then Lemmas 7 and 6 imply that uv_2v_3 is a triangle and $\deg(v_2), \deg(v_3) \geq 4$. $\square$

We are now ready to prove the main result.

Proof of Theorem 1. Suppose for a contradiction that there exists a counterexample, and let G be a minimal one. Give each vertex v charge $5 - 3\deg(v)/2$; the sum of charges is equal to $p(G)$.

Each vertex of degree three sends $1/4$ to each incident vertex of degree at least four. By Corollary 8, the final charge of a vertex of degree three is at most $1/2 - 2 \times 1/4 = 0$. A vertex v of degree at least 4 has final charge at most $(5-3\deg(v)/2)+\deg(v) \times 1/4 = 5-5\deg(v)/4 \leq 0$. Hence, all charges are non-positive. Since no charge was created or lost, the sum of charges is still equal to $p(G)$, and thus $p(G) \leq 0$. This contradicts the assumption that $p(G) \geq 3$. $\square$

1 Consequences

Corollary 9. *Let G be a graph. If $p(G') \geq 3$ for every $G' \subseteq G$, then G is 3-colorable.*

Theorem 10. *Every planar graph G of girth at least 5 is 3-colorable. Furthermore, if u and v are non-adjacent vertices of G, then there exists a 3-coloring that gives u and v the same color, as well as a 3-coloring that gives u and v different colors.*

Proof. Every planar graph H of girth at least 5 with at least four vertices satisfies $\|H\| \leq \frac{5}{3}(|H| - 2)$. Hence, $p(H) \geq 10$. If H has at most three vertices, then H is a forest, and thus $p(H) \geq 3 + 2|H|$.

Therefore, each subgraph $G' \subseteq G$ satisfies $p(G') \geq 5$, and thus G is 3-colorable by Corollary 9. If $G' \subseteq G + uv$, then G' is obtained from a planar graph of girth at least 5 by adding at most one edge, and thus $p(G') \geq 7-3 > 3$; hence, $G + uv$ is also 3-colorable, and thus G has a 3-coloring in which u and v have different colors.

Let G_1 be the graph obtained from G by identifying u with v to a new vertex w. If G_1 is not 3-colorable, then it has a 4-critical subgraph G_2, necessarily containing w. Let G_2' be the subgraph of G obtained from G_2 by un-identifying u and v. By Theorem 1, we have $p(G_2') = p(G_2) + 5 \leq 7$.

Since G_2' is planar and has girth at least 5, we conclude that $|V(G_2')| \leq 3$. But then G_2 is 3-colorable, which is a contradiction. $\square$

Theorem 11 (Grötzsch)**.** *Every planar triangle-free graph is* 3*-colorable.*

Proof. Suppose for a contradiction that G is a planar triangle-free graph that is not 3-colorable, and let us choose such a graph with $|G| + \|G\|$ minimum. Clearly, G is 4-critical.

Note that G is 2-connected. If all faces of G have length at least 5, then Euler's formula gives $p(G) \geq 10$, contradicting Theorem 1. Hence, G has a 4-face $v_1v_2v_3v_4$. Note that G cannot contain paths of length 3 both between v_1 and v_3, and between v_2 and v_4—such paths would have to intersect by planarity, resulting in a triangle. Hence, there is no such path say between v_1 and v_3. Let G' be the graph obtained from G by identifying v_1 with v_3 to a new vertex x. Note that G' is planar and triangle-free. By the minimality of G, the graph G' is 3-colorable. However, giving v_1 and v_3 the color of x turns a proper 3-coloring of G' to a proper 3-coloring of G, which is a contradiction. $\square$