NTIN 086 Selected Topics from Computational Complexity I **winter 2017/2018**

First homework assignment

Turn-in by November 1st, 2017

Problem 1. Let G be a d-regular graph on n vertices and let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix A_G. Prove that:

a) $\lambda_1 = d$ and $\lambda_n \geq -d$.

b) $\lambda_1 \neq \lambda_2$ iff G is connected.

c) For G connected, $\lambda_1 = -\lambda_n$ iff G is bipartite.

Hint: Check the appropriate eigenvectors.

Problem 2. Show that any real symmetric matrix has only real eigenvalues.

Problem 3. Let G be a hyper-cube on 2^n vertices, that is a graph with vertices $V = \{0,1\}^n$ and edges between vertices differing in exactly one coordinate.

a) Show that vectors $v_x = (v_{x,y})$, where $x, y \in \{0,1\}^n$ and $v_{x,y} = (-1)^{\sum x_i \cdot y_i}$, are the eigenvectors of the adjacency matrix of G. Determine $\lambda_2(G)$.

b) Show that $h(G) \geq 1$. *Hint:* For each pair of vertices x, y pick a directed path $\gamma_{x,y}$ between them so that each edge is used in at most 2^{n-1} paths. Conclude that that every not too big set of vertices S must have large border.

c) Using the fact $\binom{n}{n/2} \approx \sqrt{\frac{2}{\pi n}} 2^n$, find an upper-bound on $h(G)$.

Problem 4. Prove Cauchy-Schwarz inequality: For all reals $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n$

$$\left(\sum_{i=1}^{n} a_i \cdot b_i \right)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 .$$

Hint: Check $n = 2$.