

Complete graph immersions and minimum degree

Zdeněk Dvořák* Liana Yepremyan†

December 3, 2015

Abstract

An *immersion* of a graph H in another graph G is a one-to-one mapping $\varphi : V(H) \rightarrow V(G)$ and a collection of edge-disjoint paths in G , one for each edge of H , such that the path P_{uv} corresponding to the edge uv has endpoints $\varphi(u)$ and $\varphi(v)$. The immersion is *strong* if the paths P_{uv} are internally disjoint from $\varphi(V(H))$. We prove that every simple graph of minimum degree at least $11t + 7$ contains a strong immersion of the complete graph K_t . This improves on previously known bound of minimum degree at least $200t$ obtained by DeVos et al.[5]. Our result supports a conjecture of Lescure and Meyniel [9] (also independently proposed by Abu-Khazam and Langston [1]), which is the analogue of famous Hadwiger's conjecture for immersions and says that every graph without a K_t -immersion is $(t - 1)$ -colorable.

1 Introduction

In this paper, *graphs* are simple, without loops and parallel edges, while *multi-graphs* are allowed to have loops and parallel edges. In both cases, we require that the set of vertices is non-empty and finite.

A graph H is a minor of another graph G if H can be obtained from a subgraph of G by contracting edges and deleting any resulting loops and parallel edges. One of the most famous open problems in graph theory, Hadwiger's conjecture [10] from 1943 says that every loopless graph without a K_t -minor is $(t - 1)$ -colorable. This conjecture is widely open for $t \geq 7$; while $t \leq 4$ cases are trivial, $t = 5$ case is equivalent to the celebrated Four-Color Theorem and $t = 6$ case was solved by Robertson, Seymour and Thomas [15]. Note that a stronger conjecture by Hajós was proposed in 1940's [12]. A graph H is a *topological minor* of another graph G if a subgraph of G can be obtained from H by subdividing some edges. Hajós conjectured that every graph without a K_t -topological minor must be $(t - 1)$ -colorable. However, this is known to be

*Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported in part by (FP7/2007-2013)/ERC Consolidator grant LBCAD no. 616787.

†School of Computer Science, McGill University, Montreal, Canada. E-mail: liana.yepremyan@mail.mcgill.ca.

false in general; Catlin [3] disproved this conjecture for all $t \geq 7$, for $t \leq 4$ it follows from the results of Dirac [8] and $t = 5, 6$ cases are still open. The topic of this paper is inspired by the immersion variant of Hadwiger's conjecture proposed by Lescure and Meyniel [9] (and later, independently, by Abu-Khzam and Langston [1]).

An *immersion* of a graph H in a graph G is a one-to-one mapping $\varphi : V(H) \rightarrow V(G)$ and a collection of edge-disjoint paths in G , one for each edge of H , such that the path P_{uv} corresponding to the edge uv has endpoints $\varphi(u)$ and $\varphi(v)$. The vertices in $\varphi(V(H))$ are called *branch vertices*. We also give an alternative definition. We define the operation of *splitting off* pairs of edges as follows. A pair of distinct adjacent edges uv and vw is split off from their common vertex v by deleting the edges uv and vw and adding the edge uw (note that this might result in a parallel edge or a loop). We say that G contains an immersion of the graph H if a graph isomorphic to H can be obtained from a subgraph of G by splitting off pairs of edges (and removing isolated vertices). Note that topological minor containment implies both minor and immersion containments, while minor and immersion containments are incomparable.

Conjecture 1 (Lescure and Meyniel [9], Abu-Khzam and Langston [1]). *Every graph without a K_t -immersion is $(t - 1)$ -colorable.*

Conjecture 1 is known to be true for $t \leq 7$; for $t \leq 4$ the arguments are trivial, for $5 \leq t \leq 7$ it was proven by DeVos et al.[6] and independently, by Lescure and Meyniel [9] (their proof of $t = 7$ case is not published). In all mentioned cases the authors actually prove stronger statements, showing that only a lower bound on the minimum degree is required to ensure the existence of the immersion.

Let $f(t)$ be the smallest value such that every graph of minimum degree at least $f(t)$ contains an immersion of K_t . The idea of considering this function $f(t)$ was first proposed in [6], as the natural analogue of classical results showing that large average degree (equivalently, large minimum degree) in a graph implies a K_t -minor or K_t -topological minor containment. To this end, it is known that average degree $\Omega(t\sqrt{\log t})$ in a graph forces a K_t -minor and this bound is tight (Kostochka [14] and Thomason [16]). Similarly, average degree $\Omega(t^2)$ forces a topological minor of K_t , as proved independently by Bollobás and Thomason [2] and by Komlós and Szemerédi [13]; again, the bound is tight.

For immersions, it is easy to see that $f(t) \geq t - 1$. In [6] and [9], the authors proved that $f(t) = t - 1$ when $t = 5, 6, 7$; this implies Conjecture 1, since every t -chromatic graph has a subgraph of minimum degree at least $t - 1$. However, an example due to Paul Seymour showed that this is not true in general; the graph obtained from the complete graph K_{12} by removing edges of four disjoint triangles does not contain an immersion of K_{10} . DeVos et al. [5] generalized this construction, giving graphs of minimum degree $t - 1$ and no K_t -immersion for $t = 10$ and $t \geq 12$. Collins and Heenehan [4] found infinite families of such examples for all $t \geq 8$.

Nevertheless, we are not aware of any construction showing a larger gap between $f(t)$ and t . Let us remark that if $f(t) \leq t$ for some integer t , then

Conjecture 1 holds for t , see [1]. Hence, we would like to pose the following problem.

Problem 2. *Is it true that for every positive integer t , all graphs of minimum degree at least t contain K_t as an immersion?*

DeVos et al. [5] gave the first non-trivial upper bound on $f(t)$, showing that $f(t) \leq 200t$. Actually, they proved this even for *strong immersions*, when paths P_{uv} are internally disjoint from $\varphi(V(H))$, i.e., the paths P_{uv} are not allowed to use the branch vertices except as endpoints. They also showed that even stronger statement is true for dense graphs; every graph with $\Omega(n^2)$ edges contains an immersion of a clique of linear size such that every path of the immersion has exactly one internal vertex. In this paper we improve on their bound on $f(t)$ as follows.

Theorem 3. *For every positive integer t , if G is a graph with minimum degree at least $11t + 7$, then it contains K_t as a strong immersion.*

The rest of the paper is occupied by the proof of this result. In Section 2 we present all the preliminary results while Section 3 contains the main proof of Theorem 3.

2 Preliminary results

In this section we present all the auxiliary results necessary for our proof of Theorem 3.

DeVos et al. [5] observed that the complete bipartite graph $K_{t,t}$ contains K_t as a strong immersion (in fact, a slightly more involved argument shows that $K_{t-1,t-1}$ contains K_t as a strong immersion, which is the best possible, since a graph containing an immersion of K_t must have at least t vertices of degree at least $t - 1$). We will use two generalizations of this claim. Before we state them, let us recall a result on list edge coloring.

Theorem 4 (Häggkvist and Janssen [11]). *For every $n \geq 1$, the line graph of K_n has list chromatic number at most n .*

Lemma 5. *Let t be a positive integer. Let A and B be disjoint sets of vertices of G , with $|A| \geq t$. If each two non-adjacent vertices in A have at least t common neighbors in B , then G contains K_t as a strong immersion, with all branch vertices contained in A .*

Proof. Let $A_0 = \{v_1, \dots, v_t\}$ be a subset of A of size t . Let H be the complement of $G[A_0]$. By Theorem 4, there exists a proper edge coloring $\varphi : E(H) \rightarrow B$ such that $\varphi(uv)$ is a common neighbor of u and v for each $uv \in E(H)$. For each $uv \in E(H)$, split off in G the pair $u\varphi(uv), v\varphi(uv)$ of edges, obtaining the edge uv . This results in a graph strongly immersed in G such that A_0 induces a clique. \square

Lemma 6. *Let t be a positive integer. Every complete multipartite graph G of minimum degree at least t contains K_t as a strong immersion.*

Proof. We prove the claim by induction on t . For $t = 1$, the claim is obviously true.

Let V_1, \dots, V_k be the parts of G . Note that since the minimum degree of G is at least t , we have $|V_2| + \dots + |V_k| \geq t$. Let $s = |V_1|$. If $s \geq t$, then G contains $K_{t,t}$ as a subgraph, and thus G contains a strong immersion of K_t . Suppose that $s \leq t - 1$. By the induction hypothesis, $G - V_1$ contains K_{t-s} as a strong immersion θ_1 . Since $|V(G - V_1)| \geq t$, there exists a set $B \subset V(G - V_1)$ of size s that does not contain any branch vertex of θ_1 . By Lemma 5, the complete bipartite graph between V_1 and B contains a strong immersion θ_2 of K_s , with all branch vertices contained in V_1 . Then, θ_1 together with θ_2 and edges between V_1 and $V(G - V_1) \setminus B$ form a strong immersion of K_t in G . \square

Let us remark that the lower bound on minimum degree in Lemma 6 cannot be improved in general, since e.g. the complete 4-partite graph with parts of size three is known [5] not to immerse K_{10} .

A graph G is *hypomatchable* if $G - v$ has a perfect matching for every $v \in V(G)$. We will need Edmonds-Gallai theorem on maximum matchings in the following form (see e.g. Diestel [7], Theorem 2.2.3).

Theorem 7 (Edmonds-Gallai). *Every graph G contains a set $T \subseteq V(G)$ and a matching M of size $|T|$ such that each component of $G - T$ is hypomatchable, each edge of M has exactly one end in T , and no two edges of M have end in the same component of $G - T$.*

Graphs without a strong immersion of K_t whose complement neither has a perfect matching nor is hypomatchable have a special structure, as shown in the following lemma; a somewhat weaker form of this result appears implicitly in [5].

Lemma 8. *Let t be a positive integer. Let G be a graph with n vertices that does not contain a complete multipartite subgraph with minimum degree at least t . Suppose that the complement \overline{G} of G neither has a perfect matching nor is hypomatchable, and let $T \subseteq V(G)$ be as in Theorem 7 applied to \overline{G} . There exists a non-empty set $W \subseteq V(G) \setminus T$ such that $|T| \leq |W| \leq t - 1$ and each vertex of W has degree at least $n - t$ and is adjacent in G to all vertices of $V(G) \setminus (T \cup W)$.*

Proof. Let C_1, \dots, C_k be the components of $\overline{G} - T$. Since the complement of G neither has a perfect matching nor is hypomatchable, we have $k \geq 2$ and $k > |T|$. Note that G contains a complete multipartite subgraph G' with parts $V(C_1), \dots, V(C_k)$, and by the assumptions, its minimum degree is less than t . By symmetry, we can assume that vertices of $V(C_k)$ have degree at most $t - 1$ in G' , and thus $|V(C_1)| + \dots + |V(C_{k-1})| \leq t - 1$. Let $W = \bigcup_{i=1}^{k-1} V(C_i)$.

Note that $|W| \geq k - 1 \geq |T|$. Each vertex of W is adjacent to all vertices of $V(C_k) = V(G) \setminus (T \cup W)$. Also, it is adjacent to all vertices of all but one component of $\overline{G} - T$ contained in W , i.e., it has degree at least $k - 2 \geq |T| - 1$

in $G[W]$. Hence, each vertex of W has degree at least $n - |T| - |W| + (|T| - 1) = n - |W| - 1 \geq n - t$. \square

As a first step towards proving Theorem 3, we aim to reduce the problem to Eulerian graphs (i.e., graphs with only even degree vertices); this is convenient, since even degree vertices can be completely split off in the process of finding an immersion. DeVos et al. [5] showed that a graph of minimum degree at least $2d$ contains an Eulerian subgraph of minimum degree at least d . To avoid losing half of the degree in this preprocessing step, we use a somewhat more involved construction (Lemma 13) that allows us to only decrease the minimum degree by 6.

To eliminate vertices of odd degree, we apply the following well-known fact.

Lemma 9. *Let T be a tree, and let $f : V(T) \rightarrow \{0, 1\}$ be arbitrary function such that $\sum_{v \in V(T)} f(v)$ is even. Then there exists a forest $T' \subseteq T$ such that every vertex v satisfies $\deg_{T'} v \equiv f(v) \pmod{2}$.*

Hence, to achieve our goal it would suffice to find a spanning tree with bounded maximum degree. A sufficient condition for the existence of such a spanning tree was found by Win [17]. Let $c(G)$ denote the number of components of a graph G .

Theorem 10 (Win [17]). *Let $k \geq 2$ be an integer. If every $S \subseteq V(G)$ satisfies $c(G - S) \leq (k - 2)|S| + 2$, then G has a spanning tree of maximum degree at most k .*

To apply this result, we need to deal with graphs such that removal of a small number of vertices creates many components; such graphs either have small connectivity or contain vertices of large degree, and the two following lemmas address these possibilities. In a graph G , for any $X \subseteq V(G)$ we denote by ∂X the set of edges having exactly one endpoint in X .

Lemma 11. *For every even positive integer d and a graph G of minimum degree at least d , there exists $X \subseteq V(G)$ such that $|\partial X| < d$ and $G[X]$ is $(d/2)$ -edge-connected.*

Proof. The claim is trivial if G is d -edge-connected. Otherwise, let X be a smallest non-empty set of vertices of G such that $|\partial X| < d$. If $G[X]$ were not $(d/2)$ -edge-connected, there would exist a partition of X to non-empty subsets X_1 and X_2 such that there are less than $d/2$ edges with one end in X_1 and the other end in X_2 . However, then $|\partial X_1| + |\partial X_2| < |\partial X| + 2 \cdot \frac{d}{2} < 2d$, and thus either $|\partial X_1| < d$ or $|\partial X_2| < d$, contradicting the minimality of X . \square

Lemma 12. *Let t and d be positive integers. If G is a graph of minimum degree at least d and G does not contain K_t as a strong immersion, then G contains as a strong immersion a graph of minimum degree at least $d - 1$ and maximum degree at most $d + t$.*

Proof. Without loss of generality, we can assume that for every $uv \in E(G)$, either $\deg u = d$ or $\deg v = d$, as otherwise we can remove the edge uv . Let S denote the set of vertices of G of degree at least $d+t+1$, and note that S is an independent set in G , by our previous observation.

Consider any $S' \subseteq S$, and let Y be the set of vertices of G adjacent to a vertex in S' . We claim that $|Y| \geq |S'|$: indeed, the number of edges with one end in S' and the other end in Y is at least $(d+t+1)|S'|$, and at most $d|Y|$. Hence, Hall's theorem implies that there exists an injective function $g : S' \rightarrow Y$ such that $vg(v) \in E(G)$ for all $v \in S'$.

Consider each vertex $v \in S$ in turn, and let H be the subgraph induced by its neighbors. Each vertex of H has degree at most $d < |V(H)| - t$. By Lemma 8, we conclude that the complement \overline{H} of H either has perfect matching or is hypomatchable. In the former case, let M be the perfect matching in \overline{H} ; in the latter case, let M be the perfect matching in $\overline{H} - g(v)$. We remove v and add M to the edge set of G . Note that the resulting graph is strongly immersed in G .

This way, we eliminated all vertices of degree greater than $d+t$, while we only decreased degrees of vertices of $g(S)$ by one. It follows that the resulting graph has minimum degree at least $d-1$ and maximum degree at most $d+t$. \square

Lemma 13. *Let t be a positive integer and let $d \geq 2t+12$ be an even integer. If G is a graph of minimum degree at least $d+6$ that does not contain K_t as a strong immersion, then G contains as a strong immersion an Eulerian graph G' such that $\sum_{v \in V(G')} \max(0, d - \deg v) < d$.*

Proof. By Lemma 12, G contains as a strong immersion a graph G_1 of minimum degree at least $d+5$ and maximum degree at most $d+t+6$. By Lemma 11, there exists $X \subseteq V(G_1)$ such that $|\partial X| < d$ and $G_1[X]$ is $(d/2)$ -edge-connected. Let $G_2 = G_1[X]$.

We claim that G_2 has a spanning tree of maximum degree at most 5. Indeed, it suffices to verify the assumptions of Theorem 10. Consider any non-empty $S \subseteq X$. The number of edges with exactly one end in S is at most $|S|(d+t+6)$. On the other hand, G_2 is $(d/2)$ -edge-connected, and thus the number of such edges is at least $c(G_2 - S)d/2$. It follows that $c(G_2 - S) \leq \frac{2(d+t+6)}{d}|S| \leq 3|S|$.

Let T be a spanning tree of G_2 of maximum degree at most 5. By Lemma 9, there exists $T' \subseteq T$ such that the parity of the degree of each vertex is the same in T' and in G_2 . Hence, $G' = G_2 - E(T')$ is an Eulerian graph. Since G_1 has minimum degree at least $d+5$, each vertex of G' except for those incident with the edges of ∂X has degree at least d , and $\sum_{v \in V(G')} \max(0, d - \deg v) \leq |\partial X| < d$. \square

3 The Proof of Theorem 3

Let us first give a brief outline of the proof. We try to split off the vertices of the graph G one by one, preserving the degrees of all other vertices and not creating any parallel edges or loops. This is not possible if the complement of

the subgraph induced by the neighborhood of the considered vertex a does not have a perfect matching; in this case, Lemma 8 implies that G contains vertices that have many common neighbors with a . Let A denote the set of such vertices. Next, we try to split off completely all the vertices in A , and again, we only fail when some further vertices have many neighbors in common with the vertices of A . Thus, we include these vertices in A , and repeat the process. Eventually, we include at least t vertices in A , at which point Lemma 5 applies and gives a strong immersion of K_t .

This overall structure of the proof is inspired by the proof of DeVos et al. [5]. The main difference is that in their approach, they allow splitting only a part of the vertices of A , and thus the size of A may increase and decrease throughout the argument. The termination is ensured by the fact that splitting the vertices of A increases the density of the graph induced by the common neighbors of A , which eventually makes it possible to find a strong immersion of K_t using another argument specific to very dense graphs. Avoiding this step enables us to significantly lower the multiplicative constants (at the expense of a somewhat more complicated analysis of the numbers of common neighbors of vertices of A).

Definition 1. Let $t \geq 1$ and $d \geq 11t$ be integers. A (t, d) -state is a triple $T = (G, A, B)$, where G is an Eulerian graph such that $\sum_{v \in V(G)} \max(0, d - \deg v) < d$, and $A \neq \emptyset$ and B are disjoint subsets of vertices of G , satisfying the following conditions:

- (i) $d - |A| \leq |B| \leq d$, and
- (ii) there exists an ordering a_1, \dots, a_p of the elements of A such that for $1 \leq i \leq p$, the vertex a_i is adjacent to all but at most $|A| + 2i$ vertices of B .

We say that a set $A \subseteq V(G)$ is *splittable* if there exists a graph G' with vertex set $V(G) \setminus A$ that is strongly immersed in G , such that $\deg_{G'} v = \deg_G v$ for every $v \in V(G')$. A *near-matching* is a graph of maximum degree two, and its *center* is the set of its vertices of degree two. Let us now formulate the main part of our argument, making precise the claims from the first paragraph of this section.

Lemma 14. Let $t \geq 1$ and $d \geq 11t$ be integers. Let $T = (G, A, B)$ be a (t, d) -state, where G does not contain K_t as a strong immersion. If A is not splittable and $|A| \leq t - 1$, then there exists a (t, d) -state $T = (G, A', B')$ such that $A \subsetneq A'$ and $|A'| < |A| + t$.

Proof. Let a_1, \dots, a_p be the ordering of A from Definition 1. We try to completely split off a_p, a_{p-1}, \dots, a_1 in this order, obtaining a sequence $G_0 = G, G_1, \dots, G_m$ of multigraphs strongly immersed in G . We maintain the following invariants for all $i \geq 0$:

1. $V(G_i) = V(G) \setminus \{a_p, \dots, a_{p-i+1}\}$ and $E(G_i[B]) \setminus E(G)$ is a union of i near-matchings with pairwise disjoint centers; let Q_i denote the union of these centers.

2. $\deg_{G_i} v = \deg_G v$ for all $v \in V(G_i)$.
3. All loops are incident with vertices of A . Each parallel edge of G_i either has both ends in A or one end in A and multiplicity 2. Let R_i denote the set of vertices of G_i not belonging to A that are incident with such a double edge. Each vertex of R_i is incident with only one double edge, $Q_i \cap R_i = \emptyset$ and $|Q_i| + |R_i| \leq i$.

Clearly, these invariants hold for G_0 . Suppose we already constructed G_{i-1} . Let $a = a_{p-i+1}$. For each double edge joining a with a vertex of R_{i-1} , call one of the edges of the pair *primary* and the other one *secondary*. Let us define an auxiliary graph H as follows. The vertices of H are the edges of G_{i-1} incident with a such that their other ends belong to $V(G_{i-1} - A)$. Two distinct edges $e_1 = au$ and $e_2 = av$ are adjacent in H if either $uv \in E(G_{i-1})$ or $u, v \in R_{i-1}$ and at least one of e_1 and e_2 is secondary (including the case $u = v$, see Figure 1 for an illustration).

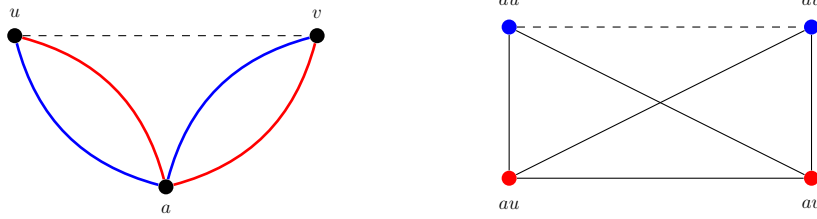


Figure 1: Primary (blue) and secondary (red) edges in G_{i-1} and the corresponding subgraph in H (assuming $uv \notin E(G_{i-1})$)

Let us first consider the case when the complement \overline{H} of H either has a perfect matching or is hypomatchable. Note that there exists $e \in V(\overline{H}) \setminus (Q_{i-1} \cup R_{i-1})$, since a has at least $|B| - (|A| + 2(p - i + 1)) \geq d - 4t > t > i$ distinct neighbors in B . If \overline{H} has a perfect matching, then let M be such a perfect matching. If \overline{H} is hypomatchable, then let M be a perfect matching in $\overline{H} - e$. Let M' be the graph with vertex set $V(G_{i-1} - A)$ and edge set $\{xy : e_1 e_2 \in M, e_1 = ax, e_2 = ay\}$.

Note that the construction of H implies that M' is a near-matching contained in the complement of G_{i-1} and that the center Q of M' is a subset of R_{i-1} . The graph G_i is obtained from G_{i-1} by splitting off a completely so that the edges from a to $V(G_{i-1} - A)$ (except for e when \overline{H} is hypomatchable) give rise to M' and the rest of edges (those to A and the edge e if \overline{H} is hypomatchable) are split off arbitrarily (it is possible to split off a completely, since it has even degree and we do not restrict parallel edges and loops with both ends in A). The double edges incident with the center of M' were split off and at most one new parallel edge with an end outside of A is created; this happens in the case when \overline{H} is hypomatchable and $e = az$ ends up to be split off to form such a parallel edge. By our original choice of e , $z \notin R_{i-1}$, thus z is incident with at most

one parallel edge in G_i and such a parallel edge has multiplicity 2. Therefore, letting $Q_i = Q_{i-1} \cup Q$, we have $R_i \subseteq (R_{i-1} \setminus Q) \cup \{z\}$, and thus $Q_i \cap R_i = \emptyset$ and $|Q_i| + |R_i| \leq |Q_{i-1}| + |R_{i-1}| + 1 \leq i$. Hence all the invariants are satisfied.

Since A is not splittable, we cannot split off all vertices of A in this way—the invariants imply that G_p would be a graph with vertex set $V(G) \setminus A$ that is strongly immersed in G , such that $\deg_{G_p} v = \deg_G v$ for every $v \in V(G_p)$. Hence, for some $i \leq p$, the graph \overline{H} neither has a perfect matching nor is hypomatchable. Note that H does not contain a complete multipartite subgraph with minimum degree at least $t + |R_{i-1}| \leq 2t$: otherwise, after removing vertices corresponding to the secondary edges we obtain a complete multipartite subgraph with minimum degree at least t which, by definition, corresponds to a complete multipartite subgraph with minimum degree at least t in G_{i-1} , in contrary to Lemma 6.

Let T and W be the sets from Lemma 8 applied to H , where $|T| \leq |W| < 2t$ and let $W' = \{w \in V(G_{i-1}) : aw \in W\}$. Clearly $W' \neq \emptyset$. We will show that (G, A', B') is a (t, d) -state for $A' = A \cup W'$ and $B' = B \setminus W'$.

First we check that $|W'| \leq t - 1$. Let $T' = \{t \in V(G_{i-1}) : at \in T\}$, $R' = R_{i-1} \cap W'$ and $R'' = R_{i-1} \setminus R'$. Note that $W' \neq \emptyset$ and since each edge in W has multiplicity at most 2, we have $|T| \leq |W| \leq |W'| + |R'|$. Furthermore, each vertex of W' is adjacent in G_{i-1} to all vertices of $(N(a) \cap B) \setminus (W' \cup T' \cup R'')$. But we have

$$\begin{aligned} |(N(a) \cap B) \setminus (T' \cup W' \cup R'')| &\geq |N(a) \cap B| - |T| - |W| - |R_{i-1}| \\ &\geq |B| - |B \setminus N(a)| - 2|W| - |R_{i-1}| \\ &\geq d - 9t > t. \end{aligned}$$

Hence if $|W'| \geq t$, then G_{i-1} contains $K_{t,t}$ as a subgraph which implies that G contains K_t as a strong immersion, a contradiction.

So to finish the proof of the lemma it remains to check conditions (i)-(ii) in Definition 1. The condition (i) holds trivially. For condition (ii), let $A' = \{a_1, \dots, a_{p+k}\}$, where a_{p+1}, \dots, a_{p+k} is an arbitrary ordering of W' . For $1 \leq j \leq p$, we trivially have $|B' \setminus N(a_j)| \leq |B \setminus N(a_j)| \leq |A| + 2j \leq |A'| + 2j$.

Now suppose $p+1 \leq j \leq p+k$. In G_{i-1} , a_j is adjacent to all vertices of $(N(a) \cap B) \setminus (W' \cup T' \cup R'')$. Thus, a_j has at most $|B \setminus N(a)| + |T'| + |R''|$ non-neighbors in B' in the graph G_{i-1} . Since $E(G_{i-1}[B]) \setminus E(G)$ is a union of $(i-1)$ near-matchings with pairwise distinct centers, G_{i-1} contains at most i edges incident with a_j that do not belong to G . Recall that $|T'| \leq |W'| + |R'|$. Hence in G we have

$$\begin{aligned} |B' \setminus N(a_j)| &\leq |B \setminus N(a)| + |T'| + |R''| + i \\ &\leq |A| + 2(p-i+1) + |W'| + |R_{i-1}| + i \\ &\leq |A| + 2(p-i+1) + |W'| + 2i - 1 \\ &= |A'| + 2p + 1 \\ &\leq |A'| + 2j, \end{aligned}$$

as desired. \square

It remains to show that there exists some (t, d) -state to which Lemma 14 can be applied, and that the number of applications of Lemma 14 is bounded. These facts follow from the next two lemmas.

Lemma 15. *Let d be a positive integer. If G is a graph such that $\sum_{v \in V(G)} \max(0, d - \deg v) < d$, then G contains a vertex of degree at least d .*

Proof. Let X be the set of vertices of degree less than d in G . The claim is trivial if $X = \emptyset$, hence assume that there exists a vertex $x \in X$. If all neighbors of x belonged to X , then since the graph G is simple, we would have $\deg x \leq \sum_{v \in V(G) \setminus \{x\}} \max(0, d - \deg v)$, and thus

$$\sum_{v \in V(G)} \max(0, d - \deg v) = d - \deg x + \sum_{v \in V(G) \setminus \{x\}} \max(0, d - \deg v) \geq d,$$

contradicting the assumptions. Therefore, x has a neighbor v not belonging to X , i.e., $\deg v \geq d$. \square

Lemma 16. *Let $t \geq 1$ and $d \geq 11t$ be integers. Let $T = (G, \{a\}, B)$ be a (t, d) -state. If G does not contain K_t as a strong immersion, then there exists a splittable set $A \subsetneq V(G)$ with $a \in A$.*

Proof. Note that by Lemma 15, G has at least $d + 1$ vertices. If G does not contain such a splittable set, then repeated applications of Lemma 14 give us a (t, d) -state (G, A', B') , where $a \in A'$ and $t \leq |A'| \leq 2t$. Let a_1, \dots, a_p be the ordering of A' as in Definition 1. For $1 \leq i < j \leq t$, we have $|B' \setminus N(a_i)| \leq |A'| + 2i \leq 4t$ and $|B' \setminus N(a_j)| \leq 4t$, and thus a_i and a_j have at least $|B'| - 8t \geq d - |A'| - 8t \geq t$ common neighbors in B' . However, by Lemma 5, this implies that G contains K_t as a strong immersion, which is a contradiction. \square

Combining these results, we now easily obtain a strong immersion of K_t as required.

Lemma 17. *Let $t \geq 1$ and $d \geq 11t$ be integers. If G is an Eulerian graph such that $\sum_{v \in V(G)} \max(0, d - \deg v) < d$, then G contains K_t as a strong immersion.*

Proof. Suppose for a contradiction that G does not contain K_t as a strong immersion, and let G be such a graph with the smallest number of vertices. By Lemma 15, there exists $v \in V(G)$ of degree at least d . Let B be a set of d neighbors of v . Then $T = (G, \{v\}, B)$ is a (t, d) -state. By Lemma 16, there exists a non-empty splittable $A \subsetneq V(G)$. Hence, there exists a graph G' with vertex set $V(G) \setminus A$ that is strongly immersed in G , such that $\deg_{G'} v = \deg_G v$ for every $v \in V(G')$, and in particular G' is Eulerian and $\sum_{v \in V(G')} \max(0, d - \deg v) < d$. However, by the minimality of G , the graph G' contains K_t as a strong immersion, which is a contradiction. \square

Proof of Theorem 3. Suppose for a contradiction that G does not contain K_t as a strong immersion. Let $d \in \{11t, 11t+1\}$ be even. By Lemma 13, G contains as a strong immersion an Eulerian graph G' such that $\sum_{v \in V(G')} \max(0, d - \deg v) < d$. However, then G' contains K_t as a strong immersion by Lemma 17, which is a contradiction. \square

References

- [1] ABU-KHZAM, F., AND LANGSTON, M. Graph coloring and the immersion order. In *Computing and Combinatorics*, T. Warnow and B. Zhu, Eds., vol. 2697 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2003, pp. 394–403.
- [2] BOLLOBÁS, B., AND THOMASON, A. Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs. *European J. Combin.* 19 (1998), 883–887.
- [3] CATLIN, P. A. A bound on the chromatic number of a graph. *Discrete Math.* 22 (1978), 81–83.
- [4] COLLINS, K. L., AND HEENEHAN, M. E. Constructing Graphs with No Immersion of Large Complete Graphs. *Journal of Graph Theory* 77 (2014), 1–18.
- [5] DEVOS, M., DVOŘÁK, Z., FOX, J., McDONALD, J., MOHAR, B., AND SCHEIDE, D. Minimum degree condition forcing complete graph immersion. *Combinatorica* 34 (2014), 279–298.
- [6] DEVOS, M., KAWARABAYASHI, K., MOHAR, B., AND OKAMURA, H. Immersing small complete graphs. *Ars Math. Contemp.* 3 (2010), 139–146.
- [7] DIESTEL, R. *Graph Theory*, third ed., vol. 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, 2005.
- [8] DIRAC, G. A property of 4-chromatic graphs and some remarks on critical graphs. *Journal of London Math Society* 27 (1952), 85–92.
- [9] F. LESCURE, H. M. On a problem upon configurations contained in graphs with given chromatic number. *Ann. Discrete Math.* 41 (1989), 325–331. Graph theory in memory of G.A. Dirac (Sandbjerg, 1985).
- [10] HADWIGER, H. Über eine Klassifikation der Streckenkomplexe. *Vierteljahrsschr. Naturforsch. Ges. Zürich* 88 (1943), 133–143.
- [11] HÄGGKVIST, R., AND JANSSEN, J. New bounds on the list-chromatic index of the complete graph and other simple graphs. *Combinatorics, Probability and Computing* 6, 03 (1997), 295–313.
- [12] HAJÓS, G. Über eine Konstruktion nicht n -farbbarer Graphen. *Wiss. Z. MartinLuther Univ. Halle-Wittenberg Math. Naturwiss. Reihe*, 10 (1961), 116–117.
- [13] KOMLÓS, J., AND SZEMERÉDI, E. Topological cliques in graphs. II. *Combin. Probab. Comput.* 5 (1996), 79–90.
- [14] KOSTOCHKA, A. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica* 4 (1984), 307–316.

- [15] ROBERTSON, N., SEYMOUR, P. D., AND THOMAS, R. Hadwiger's conjecture for K_6 -free graphs. *Combinatorica*, 13 (1993), 279–361.
- [16] THOMASON, A. The extremal function for complete minors. *J. Combin. Theory, Ser. B* 81 (2001), 318–338.
- [17] WIN, S. On a connection between the existence of k -trees and the toughness of a graph. *Graphs and Combinatorics* 5, 1 (1989), 201–205.