

## Pattern-avoiding fillings of rectangular shapes

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ABSTRACT. A *constrained table of shape*  $r \times s$  is an empty table with  $r$  rows and  $s$  columns, together with two sequences of nonnegative integers: the *row constraints*  $(x_1, \dots, x_r)$ , and the *column constraints*  $(y_1, \dots, y_s)$ . A *filling* of the table is an assignment of nonnegative integers into the cells of the table such that the entries of the  $i$ -th row sum up to  $x_i$  and the entries of the  $j$ -th column sum up to  $y_j$ . For a given  $0 - 1$ -matrix  $P$ , we say that the filling *avoids*  $P$ , if it contains no submatrix of the same size as  $P$  with strictly positive entries in all the positions that correspond to 1-entries of  $P$ .

We prove that for a permutation matrix  $P$  of order at most 3, the number of  $P$ -avoiding fillings of a given constrained table only depends on the order of  $P$  and the (unordered) multiset  $\{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_s\}$ . Thus, for example, a  $2 \times 10$  table with both row constraints equal to 5 and all column constraints equal to 1 has the same number of  $P$ -avoiding fillings as the  $6 \times 6$  table with row constraints  $(1, 1, 5, 1, 1, 1)$  and column constraints  $(1, 1, 1, 1, 5, 1)$ .

RÉSUMÉ. Un *tableau restreint de forme*  $r \times s$ , c'est un tableau vide avec  $r$  lignes et  $s$  colonnes, accompagné de deux suites de nombres entiers non négatifs, que l'on appelle *les restrictions horizontales*  $(x_1, \dots, x_r)$  et *les restrictions verticales*  $(y_1, \dots, y_s)$ . Un *remplissage* de ce tableau, c'est un placement des nombres entiers non négatifs dans les cellules du tableau, tel que la somme de cellules qui forment la  $i$ -ème ligne (ou  $i$ -ème colonne) est égale à  $x_i$  (ou  $y_i$ ). Soit  $P$  une  $0 - 1$ -matrice. On dit que le remplissage  $M$  *évite*  $P$ , si  $M$  ne contient aucune sous-matrice  $M'$  de la même forme que  $P$ , telle que les cellules de  $P$  avec un 1 correspondent aux cellules de  $M'$  avec un nombre positif.

On prouve que si  $P$  est une matrice de permutation d'ordre 3 ou moins, le nombre de remplissages d'un tableau restreint ne dépend que de l'ordre de  $P$  et de la collection  $\{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_s\}$  (qui préserve les multiplicités, mais pas l'ordre de ses éléments). Ainsi, par exemple, le tableau de forme  $2 \times 10$  avec les restrictions horizontales  $(5, 5)$  et les restrictions verticales toutes égales à 1 admet le même nombre de remplissages évitant  $P$  que le tableau de forme  $6 \times 6$  avec les restrictions  $(1, 1, 5, 1, 1, 1)$  et  $(1, 1, 1, 1, 5, 1)$ .

### 1. Introduction

In this paper, we consider the concept of pattern-avoidance in rectangular tables with prescribed row- and column-sums. This may be seen as a generalization of the popular concept of pattern-avoidance of permutations.

Let us start with basic definitions. A *constrained table of shape*  $r \times s$  is an empty table with  $r$  rows and  $s$  columns, together with two sequences of nonnegative integers: the *row constraints*  $(x_1, \dots, x_r)$ , and the *column constraints*  $(y_1, \dots, y_s)$ , satisfying

$$\sum_{i=1}^r x_i = \sum_{j=1}^s y_j.$$

A *filling* of the constrained table is a nonnegative integer matrix  $M = (m_{ij})$  with  $r$  rows and  $s$  columns, such that the sum of the entries in the  $i$ -th row is equal to  $x_i$ , and the sum of the entries in the  $j$ -th column

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is equal to  $y_j$ , formally:

$$\begin{aligned} \forall i \in 1, \dots, r: \sum_{j=1}^s m_{ij} &= x_i \\ \forall j \in 1, \dots, s: \sum_{i=1}^r m_{ij} &= y_j. \end{aligned}$$

Occasionally, we will refer to 0–1-fillings, i.e. the fillings which only take values 0 and 1.

For two sequences  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_s)$  of nonnegative integers, we let  $T[x \times y]$  denote the constrained table with row-constraints  $x$  and column-constraints  $y$ , and we let  $f(x \times y)$  denote the total number of fillings of  $T[x \times y]$ . The unordered multiset  $\{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_s\}$  will be called *the scoreline* of the table  $T[x \times y]$ .

We will adopt a convention that the rows of a table or a matrix are numbered from top to bottom and the columns are numbered from left to right.

Let  $S_n$  denote the symmetric group of order  $n$ , i.e., the group of all the permutations of the set  $[n] = \{1, 2, \dots, n\}$ . For  $\pi \in S_n$  we let  $M_\pi$  denote the permutation matrix of  $\pi$ , i.e., the  $n \times n$  matrix whose entry in row  $i$  and column  $j$  is equal to 1 if  $j = \pi(i)$  and equal to 0 otherwise. For a sequence  $x = (x_1, \dots, x_r)$  and a permutation  $\pi \in S_r$  we write  $\pi(x)$  for the sequence  $(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(r)})$ .

We now formalize our intuitive concept of pattern avoidance of fillings. A *pattern* is a matrix  $P = (p_{ij})$  with all entries equal to 0 or 1. Let  $M = (m_{ij})$  be a filling of a constrained table  $T[x \times y]$  of shape  $r \times s$ , and let  $P$  be a pattern with  $m$  rows and  $n$  columns. We say that  $M$  *contains*  $P$ , if  $M$  has a (not necessarily contiguous) submatrix  $M'$  of the same shape as  $P$  such that the following implication holds: if  $p_{ij} = 1$  for some  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , then the entry in the  $i$ -th row and  $j$ -th column of  $M'$  is positive. If  $M$  does not contain  $P$ , we say that  $M$  *avoids*  $P$ , or  $M$  is  $P$ -avoiding. We let  $f(x \times y; P)$  denote the number of  $P$ -avoiding fillings of  $T[x \times y]$ .

Note that if  $T$  is a table of shape  $r \times r$  with all the row- and column constraints equal to 1, then the fillings of  $T$  are exactly the permutation matrices of order  $r$ . Furthermore, if  $P$  is itself a permutation matrix of a permutation  $\pi \in S_n$ , then the  $P$ -avoiding fillings of  $T$  are precisely the permutation matrices corresponding to the permutations that do not contain  $\pi$  as a subpermutation. Thus, the concept of pattern-avoidance in rectangular fillings is a generalization of pattern-avoidance in permutations.

Notice that for any permutation  $\pi$  of appropriate order, we have the identity  $f(x \times y) = f(\pi(x) \times y)$ . This is because every filling  $M$  of  $T[x \times y]$  can be transformed into a filling of  $T[\pi(x) \times y]$  by permuting the rows of  $M$  according to the permutation  $\pi$ . Of course, this simple bijection in general does not preserve pattern avoidance. However, if  $P$  is a permutation matrix of order at most 3, not only do we have the identity  $f(x \times y; P) = f(\pi(x) \times \rho(y); P)$  for any  $\pi$  and  $\rho$ , but in fact, we can prove a stronger identity, stated in the following theorem:

**THEOREM 1.1 (Main result).** *Let  $T[x \times y]$  be a constrained table, let  $P$  be a permutation matrix of order at most 3. Then  $f(x \times y; P)$  is uniquely determined by the scoreline of  $T[x \times y]$  and the order of  $P$ .*

For example, consider the two tables  $T = T[(2, 2) \times (1, 1, 1, 1)]$  and  $T' = T[(2, 1, 1) \times (2, 1, 1)]$ . Both these tables have the same scoreline  $\{2, 2, 1, 1, 1, 1\}$ , and thus, they must have the same number of  $P$ -avoiding fillings for any permutation matrix  $P$  of order at most three. Indeed, if  $P$  has order 2, then both tables admit exactly one  $P$ -avoiding filling, and if  $P$  has order three, then all the six fillings of  $T$  are  $P$ -avoiding, and  $T'$  also has six  $P$ -avoiding fillings (as well as one filling containing  $P$ ). This example also shows that Theorem 1.1 cannot be extended to permutation patterns of order greater than 3.

The proof of our main result is organized as follows: in the next section, we collect some simple observations that deal with the cases when  $P$  has order at most 2. We also state previous results on fillings of Ferrers shapes, and show how these results imply that  $f(x \times y; P) = f(x \times y; Q)$  for any two permutation matrices  $P, Q$  order 3. These arguments imply that it suffices to prove Theorem 1.1 for a single fixed  $P$  of order 3.

In Section 3, we use the RSK algorithm together with basic results on Young tableaux to prove that if  $P$  is a diagonal matrix of any order  $k$ , then  $f(x \times y; P) = f(\pi(y) \times \rho(y); P)$ , where  $\pi$  and  $\rho$  are arbitrary permutations of appropriate order.

In Section 4, we study an operation called *corner flip*, defined as follows: let  $T[x \times y]$  be a constrained table of shape  $r \times s$ . Assume that for some  $t \leq r$  and  $u \leq s$  we have

$$\sum_{i=1}^t x_i = \sum_{j=1}^u y_j.$$

A corner flip is an operation that transforms the table  $T[x \times y]$  into a table  $T[x' \times y']$  of shape  $(t + s - u) \times (u + r - t)$ , where  $x' = (x_1, x_2, \dots, x_t, y_{u+1}, y_{u+2}, \dots, y_s)$  and  $y' = (y_1, y_2, \dots, y_u, x_{t+1}, x_{t+2}, \dots, x_r)$ . We will show that corner flips preserve the number of  $P$ -avoiding fillings, where  $P$  is the permutation matrix of the permutation (321). It is easy to see that any two tables of the same scoreline can be transformed to each other by a sequence of row permutations, column permutations and corner flips. Combining these facts, we obtain the proof of Theorem 1.1.

In the last section, we present some remarks on the connection between the fillings of rectangular shapes, the pattern-avoidance in permutations, and other related concepts.

## 2. Basic observations and previous results

Let us first deal with the values of  $f(x \times y; P)$  when  $P$  is a permutation matrix of order at most 2. In the trivial case when  $P$  has order 1, we see that  $f(x \times y; P) = 0$  unless all the components of  $x$  and  $y$  are zero, in which case  $f(x \times y; P) = 1$ . Let us now turn to the slightly less trivial case of permutation matrices of order 2:

LEMMA 2.1. *For every permutation matrix  $P$  of order 2 and any constrained table  $T = T[x \times y]$ , we have  $f(x \times y; P) = 1$ .*

PROOF. It suffices to prove the lemma for  $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , the other case is analogous. Let  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_s)$ . We proceed by induction on  $r + s$ . If  $r = 1$  or  $s = 1$ , the claim is clear. Otherwise, let  $k = \min\{x_r, y_s\}$ . Observe that if  $M = (m_{ij})$  is a  $P$ -avoiding filling of  $T[x \times y]$ , then  $m_{rs} = k$ , otherwise both the last row and the last column of  $M$  would contain a positive entry other than  $m_{rs}$ , and these two entries would form the forbidden pattern  $P$ . Assume now that  $k = x_r$  (the case  $k = y_s$  is symmetric). For any  $P$ -avoiding filling  $M$  of  $T$ , the last row of  $M$  is equal to  $(0, \dots, 0, k)$ . Furthermore, the remaining rows of  $M$  form a  $P$ -avoiding filling of  $T' = T[(x_1, \dots, x_{r-1}) \times (y_1, \dots, y_{s-1}, y_s - k)]$ . By the induction hypothesis, there is exactly one  $P$ -avoiding filling of  $T'$ , and adding a row  $(0, \dots, 0, k)$  to the bottom of this filling produces a  $P$ -avoiding filling of  $T$ .  $\square$

It remains to prove Theorem 1.1 for permutation patterns of order 3. As a first step, we will show that  $f(x \times y; P) = f(x \times y; Q)$  for any two patterns  $P, Q$  of order three. This result follows directly from a more general theorem by de Mier [4], which deals with fillings of arbitrary shapes. To state the result, we first need some more definitions and notation.

Let  $I_d$  be the identity matrix of order  $d$ , and let  $J_d$  be its mirror image, i.e.,  $J_d$  is the permutation matrix of the permutation  $(d, d - 1, \dots, 1)$ . For a  $k \times l$  matrix  $X$  and an  $m \times n$  matrix  $Y$  we write  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  for the  $(k + m) \times (l + n)$  matrix whose first  $k$  rows and  $l$  columns form the submatrix  $X$ , the last  $m$  rows and  $n$  columns form  $Y$  and all the other entries are zero. The matrix  $\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$  is defined similarly.

A *partition of  $n$*  is a weakly decreasing sequence of nonnegative integers  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$  whose sum is  $n$ . A partition  $\lambda$  may be conveniently represented by a *Ferrers shape*, which is a left-justified array whose  $i$ -th row has  $\lambda_i$  cells. A *constrained Ferrers shape* is a Ferrers shape whose every row and every column has a prescribed sum. We write  $T_\lambda[x \times y]$  for the Ferrers shape determined by the partition  $\lambda$ , with row constraints  $x$  and column constraints  $y$ . The fillings of constrained Ferrers shapes are a straightforward generalization of the fillings of rectangular tables. For a given (rectangular) pattern  $P$ , we say that a filling  $M$  of  $T_\lambda[x \times y]$  contains  $P$ , if  $M$  has a rectangular submatrix  $M'$  fully contained inside the shape  $\lambda$  such that  $M'$  contains  $P$  in the sense defined in the introduction. If the shape  $\lambda$  is a rectangle, then this notion of containment coincides with our previous definition. We will write  $f_\lambda(x \times y)$  for the number of fillings of  $T_\lambda[x \times y]$ , and  $f_\lambda(x \times y; P)$  for the number of these fillings that avoid  $P$ .

Constrained fillings of general shapes have been considered by Backelin, West, and Xin [1] and Stankova and West [5]; the two papers studied the case when all the constraints are equal to 1. Some of their results were subsequently extended by de Mier [4] to fillings of arbitrary shapes with arbitrary constraints. From

her paper, we shall use the following theorem, which is a generalization of a result by Backelin, West and Xin [1]:

**THEOREM 2.2** (de Mier [4]). *For every positive integer  $n$  and every two integers  $i, j$  with  $0 \leq i, j \leq n$  and every constrained Ferrers shape  $T_\lambda[x \times y]$ , the number of fillings of  $T_\lambda[x \times y]$  that avoid  $\begin{pmatrix} J_j & 0 \\ 0 & I_{n-j} \end{pmatrix}$  is equal to the number of fillings of  $T_\lambda[x \times y]$  that avoid  $\begin{pmatrix} J_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$ .*

**COROLLARY 2.3.** *For any constrained rectangular table  $T[x \times y]$  and any two permutation matrices  $P, Q$  of order 3, we have  $f(x \times y; P) = f(x \times y; Q)$ .*

**PROOF.** For  $P, Q$  chosen among  $I_3, J_3$  and  $\begin{pmatrix} J_2 & 0 \\ 0 & I_1 \end{pmatrix}$ , the claim is a special case of Theorem 1. For the other cases, we can easily establish the required identity by exploiting the symmetries of the rectangle; take, e.g.,  $P = \begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}$  and  $Q = \begin{pmatrix} J_2 & 0 \\ 0 & I_1 \end{pmatrix}$ : let us write  $x = (x_1, \dots, x_r)$  and let us define  $\bar{x} = (x_r, x_{r-1}, \dots, x_1)$ . Clearly,

$$f(x \times y; P) = f(\bar{x} \times y; Q) = f(\bar{x} \times y; I_3) = f(x \times y; J_3) = f(x \times y; Q).$$

The remaining cases are settled similarly.  $\square$

Let us stress that Corollary 2.3 does not generalize to fillings of arbitrary Ferrers shapes. In fact, apart from the three patterns  $I_3, J_3$  and  $\begin{pmatrix} J_2 & 0 \\ 0 & I_1 \end{pmatrix}$ , no other two permutations of order 3 are equinumerous for general fillings of general shapes. (If we restrict ourselves to fillings of arbitrary shapes with unit constraints only, we obtain one other equinumerous pair, namely  $\begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & I_2 \\ I_1 & 0 \end{pmatrix}$ , see [5]. More on this in Section 5.)

### 3. The RSK algorithm

In this section, we will prove the following result:

**PROPOSITION 3.1.** *Let  $T[x \times y]$  be a constrained table of shape  $r \times s$ . For every  $\pi \in S_r$  and  $\rho \in S_s$ , and for every positive integer  $n$ , we have  $f(x \times y; J_n) = f(\pi(x) \times \rho(y); J_n)$ .*

Of course, in the statement of the proposition, we could have used any other matrix from Theorem 2.2. Our choice of  $J_n$  is purely a matter of convenience.

Proposition 3.1 is an easy consequence of known results on Young tableaux and the Robinson–Schensted–Knuth (or RSK) algorithm. We will state the necessary results without proof; a useful presentation of several variants of the RSK algorithm and their relation to pattern-avoidance in fillings can be found in [3]. The proof of the basic properties of Young tableaux and the RSK correspondence can be found in textbooks of combinatorics, such as [2] or [6].

We first state the necessary definitions:

A *Young tableau*, or more verbosely, a *column-strict semi-standard Young tableau*, is a filling of a Ferrers shape such that the elements of every row form a weakly increasing sequence and the elements of every column form a strictly increasing sequence.

The *content* of a Young tableau  $P$  is a sequence  $\mu = (\mu_1, \dots, \mu_k)$  where  $\mu_i$  is the number of cells of  $P$  that contain the number  $i$ . The number of Young tableaux of shape  $\lambda$  and content  $\mu$  is known as the Kostka number, denoted  $K_{\lambda, \mu}$ .

The proof of the following standard fact can be found e.g. in [2]:

**FACT 3.2.** *Let  $\lambda$  be a partition of  $n$  and let  $\mu = (\mu_1, \dots, \mu_k)$  be a sequence of nonnegative numbers whose sum is  $n$ , let  $\pi$  be a permutation of order  $k$ . For  $\mu' = \pi(\mu) = (\mu_{\pi(1)}, \dots, \mu_{\pi(k)})$ , we have the identity  $K_{\lambda, \mu} = K_{\lambda, \mu'}$ .*

Let  $g_\pi$  be a bijection that transforms a Young tableau  $P$  of content  $\mu$  to a Young tableau  $g_\pi(P)$  of the same shape and of content  $\pi(\mu)$ .

We now summarize the needed properties of the RSK algorithm:

**FACT 3.3.** *The RSK algorithm constructs a bijection between fillings of  $T[x \times y]$  and ordered pairs of Young tableaux  $(P, Q)$  such that  $P$  and  $Q$  have the same shape,  $P$  has content  $x$  and  $Q$  has content  $y$ . Furthermore, the filling avoids  $J_n$  if and only if  $P$  and  $Q$  have less than  $n$  rows.*

These facts immediately imply Proposition 3.1:

PROOF OF PROPOSITION 3.1. Let  $x, y, \pi, \rho$  be as in Proposition 3.1. The  $J_n$ -avoiding fillings of  $T[x \times y]$  are mapped by the RSK algorithm to pairs of Young tableaux  $(P, Q)$  of the same shape  $\lambda$  of at most  $n - 1$  rows, where the content of  $P$  is  $x$  and the content of  $Q$  is  $y$ . This pair may be transformed into a pair of tableaux  $(g_\pi(P), g_\rho(Q))$  of shape  $\lambda$  and content  $\pi(x)$  and  $\rho(y)$ . By the RSK algorithm, such pairs correspond to  $J_n$ -avoiding fillings of  $T[\pi(x) \times \rho(y)]$ .  $\square$

We remark that the bijection established above does not, in general, preserve the multiset of the entries used in the corresponding fillings. In particular, it does not send 0–1-filling onto 0–1-fillings. This cannot be avoided, because, for example,  $T[(2, 1, 1) \times (2, 1, 1)]$  has no 0–1  $J_2$ -avoiding filling, while  $T[(1, 2, 1) \times (2, 1, 1)]$  has one such filling.

#### 4. The corner flip

Let us fix  $x = (x_1, \dots, x_r)$  and  $y = (y_1, \dots, y_s)$  such that  $\sum_{i=1}^r x_i = \sum_{j=1}^s y_j$ . Let us also fix  $t \leq r$  and  $u \leq s$  such that  $\sum_{i=1}^t x_i = \sum_{j=1}^u y_j$ . Recall that a corner flip is an operation that transforms a table  $T[x \times y]$  into a table  $T[x' \times y']$ , where  $x' = (x_1, x_2, \dots, x_t, y_{u+1}, y_{u+2}, \dots, y_s)$  and  $y' = (y_1, y_2, \dots, y_u, x_{t+1}, x_{t+2}, \dots, x_r)$ . We prove the following proposition:

PROPOSITION 4.1. *With the notation as above,  $f(x \times y; J_3) = f(x' \times y'; J_3)$ .*

We introduce the following terminology: let  $M$  be a matrix with at least  $t$  rows and at least  $u$  columns. The *north-west corner* of  $M$ , denoted by  $M_{\text{NW}}$ , is the submatrix of  $M$  formed by the intersection of the first  $t$  rows with the first  $u$  columns. Similarly,  $M_{\text{NE}}$  denotes the *north-east corner* of  $M$ , which is the intersection of the first  $t$  rows of  $M$  with the columns of index greater than  $u$ . The south-east and south-west corners of  $M$  are defined analogously. Thus, a matrix  $M$  of shape  $r \times s$  can be expressed as  $M = \begin{pmatrix} M_{\text{NW}} & M_{\text{NE}} \\ M_{\text{SW}} & M_{\text{SE}} \end{pmatrix}$ . Notice that if  $M$  is a filling of  $T[x \times y]$ , then the sum of the entries of  $M_{\text{NE}}$  is equal to the sum of the entries of  $M_{\text{SW}}$ . The rows of  $M$  with indices  $1, \dots, t$  are called the *northern rows*, the rows with indices greater than  $t$  are the *southern rows*, and similarly for the eastern and western columns.

Let  $(X, Y)$  be a pair of matrices. We say that a matrix  $M$  *completes  $X$  and  $Y$  inside  $T[x \times y]$*  if  $M$  is a  $J_3$ -avoiding filling of  $T[x \times y]$  with  $M_{\text{NW}} = X$  and  $M_{\text{SE}} = Y$ . The following two lemmas immediately imply Proposition 4.1.

LEMMA 4.2. *For any pair of matrices  $(X, Y)$ , there is at most one  $M$  that completes  $(X, Y)$  inside  $T[x \times y]$ .*

LEMMA 4.3. *A pair of matrices  $(X, Y)$  can be completed inside  $T[x \times y]$  if and only if the pair  $(X, Y^T)$  can be completed inside  $T[x' \times y']$ , where  $Y^T$  denotes the transpose of  $Y$ .*

By these lemmas, there is a bijection  $\phi$  that maps a  $J_3$ -avoiding filling  $M$  of  $T[x \times y]$  to the  $J_3$ -avoiding filling  $\phi(M) = M'$  of  $T[x' \times y']$  uniquely determined by the condition  $M'_{\text{NW}} = M_{\text{NW}}$  and  $M'_{\text{SE}} = M_{\text{SE}}^T$ . The existence of such a bijection implies Proposition 4.1. It remains to prove the two lemmas.

PROOF OF LEMMA 4.2. It is enough to prove that if  $M$  is a  $J_3$ -avoiding filling of  $T[x \times y]$  then both  $M_{\text{SW}}$  and  $M_{\text{NE}}$  avoid  $J_2$ . By Lemma 2.1, a  $J_2$ -avoiding matrix is uniquely determined by its row sums and column sums; in particular,  $M_{\text{NE}}$  and  $M_{\text{SW}}$  are determined by  $x, y$  and the two matrices  $M_{\text{SE}}$  and  $M_{\text{NW}}$ .

Assume that  $M$  is a  $J_3$ -avoiding filling of  $T[x \times y]$  and  $M_{\text{SW}}$  contains  $J_2$ . Since the sum of entries of  $M_{\text{SW}}$  is equal to the sum of the entries of  $M_{\text{NE}}$ , we know that  $M_{\text{NE}}$  contains at least one positive entry. This positive entry and the occurrence of  $J_2$  inside  $M_{\text{SW}}$  form the forbidden pattern  $J_3$ , which is a contradiction, showing that  $M_{\text{SW}}$  avoids  $J_2$ . By the same argument, we obtain that  $M_{\text{NE}}$  avoids  $J_2$  as well.  $\square$

Before we present the proof of Lemma 4.3, we state and prove a lemma that characterizes the pairs  $(X, Y)$  that can be completed inside  $T[x \times y]$ . We will say that a pair of matrices is *plausible* for  $T[x \times y]$ , if  $X$  and  $Y$  both avoid  $J_3$ ,  $X$  has shape  $t \times u$ ,  $Y$  has shape  $(r - t) \times (s - u)$ , and the row sums and column sums of the matrix  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  do not exceed the corresponding constraints  $x$  and  $y$ .

LEMMA 4.4. *Let  $(X, Y)$  be a pair of matrices, let  $M_0 = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ . Let  $\bar{x}_i$  be the sum of the  $i$ -th row of  $M_0$  and  $\bar{y}_j$  the sum of its  $j$ -th column. We say that the  $i$ -th row (or  $j$ -th column) is saturated if  $x_i = \bar{x}_i$  (or  $y_j = \bar{y}_j$ ). The pair  $(X, Y)$  can be completed inside  $T[x \times y]$  if and only if the following conditions are satisfied:*

- (a)  $(X, Y)$  is plausible.
- (b)  $\sum_{i=1}^t (x_i - \bar{x}_i) = \sum_{j=u+1}^s (y_j - \bar{y}_j)$  (which is equivalent to  $\sum_{j=1}^u (y_j - \bar{y}_j) = \sum_{i=t+1}^r (x_i - \bar{x}_i)$ .)
- (c) Let  $i_N$  be the largest index of a northern row of  $M_0$  such that for every  $i < i_N$ , the  $i$ -th row is saturated (in other words,  $i_N$  is the northernmost unsaturated row, or  $i_N = t$  if all northern rows are saturated). Similarly, let  $j_W$  be the largest index of a western column such that for every  $j < j_W$ , the  $j$ -th column is saturated. The submatrix of  $M_0$  induced by the rows  $\{i_N + 1, \dots, t\}$  and columns  $\{j_W + 1, \dots, u\}$  has all entries equal to 0.
- (d) With  $i_N$  and  $j_W$  as above, the submatrix of  $M_0$  induced by the rows  $\{1, \dots, i_N\}$  and columns  $\{j_W + 1, \dots, u\}$  avoids  $J_2$ . The submatrix induced by the rows  $\{i_N + 1, \dots, t\}$  and columns  $\{1, \dots, j_W\}$  avoids  $J_2$  as well.
- (e) Let  $i_S$  be the smallest row-index of a southern row such that for every  $i > i_S$ , the  $i$ -th row is saturated. Similarly, let  $j_E$  be the smallest column index of an eastern column such that for every  $j > j_E$ , the  $j$ -th column is saturated. The submatrix of  $M_0$  induced by the rows  $\{t + 1, \dots, i_S - 1\}$  and columns  $\{u + 1, \dots, j_E - 1\}$  has all entries equal to 0.
- (f) With  $i_S$  and  $j_E$  as above, the submatrix of  $M_0$  induced by the rows  $\{t + 1, \dots, i_S - 1\}$  and columns  $\{j_E, \dots, s\}$  avoids  $J_2$ . The submatrix induced by the rows  $\{i_S, \dots, r\}$  and columns  $\{u + 1, \dots, j_E - 1\}$  avoids  $J_2$  as well.

PROOF. We first show that the conditions are necessary. This is obvious in the case of (a) and (b). Assume that  $M$  completes  $X$  and  $Y$  in  $T[x \times y]$ . Assume, for contradiction, that condition (c) does not hold. Then  $M$  has a positive entry  $m_{ij} > 0$  with  $i_N < i \leq t$  and  $j_W < j \leq u$ . Since  $i_N$  is smaller than  $t$ , it is unsaturated, otherwise we would get a contradiction with  $i_N$ 's maximality. Thus,  $M$  has at least one positive entry in row  $i_N$  and an eastern column. Similarly,  $M$  has a positive entry in column  $j_W$  and a southern row. These three positive entries form the forbidden pattern  $J_3$ .

Assume now, that condition (d) fails. If the submatrix induced by the rows  $1, \dots, i_N$  and columns  $\{j_W + 1, \dots, u\}$  contains  $J_2$ , it means that  $j_W < u$  and  $j_W$  is unsaturated. Hence,  $M$  contains a positive entry in column  $j_W$  and a southern row, creating the forbidden  $J_3$ . By an analogous argument, there is no  $J_2$  in the submatrix formed by rows  $\{i_N + 1, \dots, t\}$  and columns  $\{1, \dots, j_W\}$ .

The arguments for the necessity of (e) and (f) are symmetric copies of the arguments given for the necessity of (c) and (d), respectively.

It remains to show that the conditions (a) to (f) are sufficient. Assume that  $X$  and  $Y$  satisfy these conditions. Fix  $J_2$ -avoiding matrices  $M_{NE}$  and  $M_{SW}$  in such a way that  $M = \begin{pmatrix} X & M_{NE} \\ M_{SW} & Y \end{pmatrix}$  is a filling of  $T[x \times y]$  (we do not know yet that  $M$  avoids  $J_3$ ). By condition (b) and by Lemma 2.1, we know that such  $M_{NE}$  and  $M_{SW}$  exist and are uniquely determined. By the proof of Lemma 4.2, we know that  $M$  is the only candidate for a completion of  $(X, Y)$  inside  $T[x \times y]$ . It remains to show that  $M$  avoids  $J_3$ . For contradiction, assume that  $M$  contains  $J_3$ . Fix three positive entries in  $M$  forming  $J_3$ . Assume that these entries appear in rows  $i_1 < i_2 < i_3$  and columns  $j_1 > j_2 > j_3$ . At most one of these three entries is in  $M_{SW}$  and at most one is in  $M_{NE}$ , because these two corners avoid  $J_2$  by construction. It follows that the copy of  $J_3$  intersects either  $X$  or  $Y$ . It cannot intersect both  $X$  and  $Y$  due to their mutual position. Assume that the copy of  $J_3$  intersects  $X$  (the other case is symmetric). Then clearly  $i_2 \leq t$  and  $j_2 \leq u$ , so the middle entry of the copy of  $J_3$  appears inside  $X$ . However, it is not possible to have all the three entries inside  $X$  (because  $(X, Y)$  is plausible and thus  $X$  avoids  $J_3$ ), so we may assume, losing no generality, that row  $i_1$  and column  $j_1$  intersect in  $M_{NE}$ . It follows that  $i_1$  is not saturated, which means that  $i_N \leq i_1 < i_2$ . If row  $i_3$  and column  $j_3$  intersect inside  $M_{SW}$ , we similarly obtain  $j_W \leq j_3 < j_2$  contradicting condition (c). On the other hand, if row  $i_3$  and column  $j_3$  intersect inside  $X$ , then we have a contradiction with condition (c) or (d).  $\square$

With the characterization of the matrix pairs  $(X, Y)$  that can be completed inside  $T[x \times y]$ , the proof of Lemma 4.3 is easy:

PROOF OF LEMMA 4.3. It suffices to check that a pair  $(X, Y)$  satisfies the conditions of Lemma 4.4 with respect to  $T[x \times y]$  if and only if the pair  $(X, Y^T)$  satisfies these conditions with respect to  $T[x' \times y']$ . This is obvious for conditions (a) and (b). For the remaining four conditions, we may observe that a saturated northern row or western column of  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  remains saturated in  $\begin{pmatrix} X & 0 \\ 0 & Y^T \end{pmatrix}$ . Similarly, a saturated southern row of index  $t + i$  in  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  corresponds to a saturated eastern column of index  $u + i$  in  $\begin{pmatrix} X & 0 \\ 0 & Y^T \end{pmatrix}$  and vice versa. Combining this with the observation that transposition preserves copies of  $J_2$ , we see that the last

four conditions of Lemma 4.4 are unaffected by the transition from  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  to  $\begin{pmatrix} X & 0 \\ 0 & Y^T \end{pmatrix}$  and from  $T[x \times y]$  to  $T[x' \times y']$ .  $\square$

Let us now recall and prove the main result:

**THEOREM 4.5** (Theorem 1.1, again). *Let  $T[x \times y]$  be a constrained table, let  $P$  be a permutation matrix of order at most 3. Then  $f(x \times y; P)$  is uniquely determined by the scoreline of  $T[x \times y]$  and the order of  $P$ .*

**PROOF.** We have observed earlier that the result is easy for matrices of order at most 2 (see Lemma 2.1 and the preceding discussion). Thanks to Corollary 2.3, we only need to prove the theorem for a single permutation matrix  $P$  of order 3. Our matrix of choice is the  $J_3$ . By Propositions 3.1 and 4.1, all we have to do is to realize that for any two tables  $T = T[x \times y]$  and  $T' = T[x' \times y']$  with the same scoreline, we may transform  $T$  into  $T'$  by a sequence of permutations and corner flips, which is indeed easily seen. We omit the details.  $\square$

### 5. General Remarks

We conclude the paper with some remarks and examples that put fillings of rectangular shapes into a broader context of pattern-avoidance in fillings.

Let us only consider patterns that are permutation matrices, and let us make no distinction between a permutation and its matrix.

Recall that two permutations  $\pi, \sigma$  are *Wilf-equivalent* if for every  $n$  the number of permutations of order  $n$  that avoid  $\pi$  is equal to the number of those that avoid  $\sigma$ ; in our notation, this may be written as  $f(1^n \times 1^n; \pi) = f(1^n \times 1^n; \sigma)$ , where  $1^n$  is the sequence of  $n$  ones. Let  $\pi \stackrel{1}{\sim} \sigma$  denote the fact that  $\pi$  and  $\sigma$  are Wilf-equivalent.

In [1], the authors introduce the concept of *shape-Wilf equivalence*, which can be defined as follows:  $\pi$  and  $\sigma$  are shape-Wilf equivalent (denoted by  $\pi \stackrel{1,s}{\sim} \sigma$ ) if for each Ferrers shape  $\lambda$  with  $n$  rows and  $n$  columns we have  $f_\lambda(1^n \times 1^n; \pi) = f_\lambda(1^n \times 1^n; \sigma)$ .

Allowing arbitrary constraints, we write  $\pi \sim \sigma$  if for every constrained table  $T[x \times y]$  we have  $f(x \times y; \pi) = f(x \times y; \sigma)$ , and we write  $\pi \stackrel{s}{\sim} \sigma$ , if the identity holds even for arbitrary shapes, i.e.,  $f_\lambda(x \times y; \pi) = f_\lambda(x \times y; \sigma)$  for any constrained Ferrers shape  $T_\lambda[x \times y]$ .

In general,  $\stackrel{s}{\sim}$  is different from  $\stackrel{1,s}{\sim}$ ; for example, for  $\lambda = (4, 4, 4, 3)$  we have

$$18 = f_\lambda((1, 1, 2, 1) \times (2, 1, 1, 1); \begin{pmatrix} 0 & I_2 \\ I_1 & 0 \end{pmatrix}) \neq f_\lambda((1, 1, 2, 1) \times (2, 1, 1, 1); \begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}) = 17,$$

while Stankova and West [5] have shown that  $\begin{pmatrix} 0 & I_2 \\ I_1 & 0 \end{pmatrix} \stackrel{1,s}{\sim} \begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}$ . Thus,  $\stackrel{s}{\sim}$  is a proper refinement of  $\stackrel{1,s}{\sim}$ .

As we have seen, all the permutations of order 3 are  $\sim$ -equivalent, which shows that  $\sim$  is different from  $\stackrel{1,s}{\sim}$  and  $\stackrel{s}{\sim}$ . To see that  $\sim$  is also different from  $\stackrel{1}{\sim}$ , consider the following two patterns (we omit the 0-entries for clarity):

$$P = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \bar{P} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}.$$

Clearly  $P \stackrel{1}{\sim} \bar{P}$ , since  $\bar{P}$  is a mirror image of  $P$ ; on the other hand, for  $x = (1, 1, 1, 2, 1)$  we have

$$165 = f(x \times x; P) \neq f(x \times x; \bar{P}) = 166,$$

which shows that  $P$  and  $\bar{P}$  are not  $\sim$ -equivalent. This example can also be interpreted as  $f(x \times x; P) \neq f(\bar{x} \times x; P)$ , where  $\bar{x}$  is the sequence  $x$  written backwards. This shows that Proposition 3.1 does not generalize to all forbidden patterns.

The following proposition appears in [1] (for  $\stackrel{1,s}{\sim}$ ), and [4] (for  $\stackrel{s}{\sim}$ ).

**PROPOSITION 5.1.** *Let  $A, A'$  and  $B$  be three patterns. If  $A \stackrel{1,s}{\sim} A'$ , then  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \stackrel{1,s}{\sim} \begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix}$ , and if  $A \stackrel{s}{\sim} A'$  then  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \stackrel{s}{\sim} \begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix}$ .*

In general, it is not true that  $A \sim A'$  implies  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \sim \begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix}$  (compare the example  $P$  above with  $I_4$ ); on the other hand, Proposition 5.1 easily implies that if  $A \overset{s}{\sim} A'$  and  $B \overset{s}{\sim} B'$  then  $\begin{pmatrix} A & 0 \\ 0 & \overrightarrow{B} \end{pmatrix} \sim \begin{pmatrix} A' & 0 \\ 0 & \overrightarrow{B'} \end{pmatrix}$ , where  $\overrightarrow{B}$  and  $\overrightarrow{B'}$  are the matrices obtained from  $B$  and  $B'$  by the rotation of 180 degrees (similar arguments can be made for other symmetries of the square). This allows us to obtain relations such as  $I_4 \sim \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}$ , which are not directly covered by previously mentioned theorems.

Despite these observations, a complete understanding of the  $\sim$ -equivalence, as well as of the other related concepts, seems currently out of reach.

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