Summary of the recitation on 4. 12. 2007

- Consider the following version of the Szemerédi regularity lemma: $\forall \varepsilon > 0$ $\forall m \in \mathbb{N}$ $\exists M \in \mathbb{N}$ $\exists n_0 \in \mathbb{N}$ such that every graph G with at least n_0 vertices has an ε-regular partition with at least m but not more than M parts. Show that this version is equivalent to the version given at the lecture.
- Consider the following alternative definition of ε-regular partition. For a graph $G = (V, E)$, we say that $V_1, \dots, V_k$ form an *ε-regular* partition* of G if the following conditions are satisfied:
  - The sets V_i form a disjoint partition of the vertex set V.
  - For each $i, j \leq k$, $||V_i| - |V_j|| \leq 1$.
  - For all pairs i, j, $i \neq j$, with at most εk^2 exceptions, the bipartite subgraph of G formed by all the edges between V_i and V_j is ε-regular.

  From the regularity lemma given at the lecture, deduce the following version of the regularity lemma: $\forall \varepsilon > 0$ $\forall m \in \mathbb{N}$ $\exists M \in \mathbb{N}$ $\exists n_0 \in \mathbb{N}$ such that every graph G with at least n_0 vertices has an ε-regular* partition with at least m but not more than M parts.
  You may first prove the following lemma: $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ $\exists \delta > 0$ such that if $(X \cup Y, E)$ is an ε-regular graph with $|X| = |Y| = n \geq n_0$, then any bipartite graph containing $(X \cup Y, E)$ as induced subgraph and having at most δn additional vertices is 2ε-regular.
- (Stated but not solved) Let $d \in [0, 1]$ be a constant, let $\varepsilon > 0$ be sufficiently small with respect to d. Let (X, Y) be two parts of an ε-regular bipartite graph with density d. Show that the number of pairs (x, x'), $x, x' \in X$, such that x and x' have less than $(d - \varepsilon)^2|Y|$ common neighbours in Y is at most $2\varepsilon|X|^2$.