First homework assignment
The numbers in boxes indicate the maximum number of points available for a given exercise.

1. Show that an unoriented graph $G$ has an orientation in which each vertex has at most $k$ outgoing edges if and only if each subgraph $H$ of $G$ satisfies $|E(H)| \leq k|V(H)|$.

2. For what values of $k$ is it true that every filling of the first $k$ rows of a $9 \times 9$ matrix according to the rules of sudoku can be extended into a filling of $k+1$ rows without violating the rules?

3. Let $M$ be a matrix whose entries have values 0 or 1. Show that the largest independent set of $M$ has the same cardinality as the smallest covering set of lines of $M$. An independent set of $M$ is a set of rows and columns that together cover all the positive entries of $M$.

4. Let $M$ be an $n \times n$ matrix of numbers $1, 2, \ldots, n$, where each number appears in $M$ exactly $n$ times.
   
   (a) Show that $M$ has a row or column with at least $\sqrt{n}$ different numbers.
   
   (b) Show that if $\sqrt{n}$ is an integer, then the estimate in part (a) cannot be improved.

5. Find a connected 3-regular graph with 100 vertices that has no perfect matching.

6. We say that a matching $M$ is a set of rows and columns that together cover all the positive entries of $M$. Show that if $G = (V, E)$ is a graph, let $T \subseteq V$ be a set of its vertices. $G$ has a matching that covers all the vertices from $T$ if and only if for each set $A \subseteq V$, the graph $G \setminus A$ has at most $|A| + k$ odd components.

7. Deduce Hall’s theorem from Tutte’s theorem.

8. Deduce Hall’s theorem from Gallai-Milgram’s theorem.

9. Deduce Tutte’s theorem from Hall’s theorem. For example, you may follow these steps:
   
   • Assume there is a graph which satisfies Tutte’s condition but has no perfect matching. Let $G = (V, E)$ be such a graph with the smallest possible number of vertices. Let $X \subseteq V$ be a maximal set of vertices that satisfies Tutte’s condition with equality, i.e., $G \setminus X$ has exactly $|X|$ odd components. Show that such a set $X$ exists and is nonempty.
   
   • Show that $G \setminus X$ has no even components.
   
   • Show that if $C$ is a component of $G \setminus X$ and $v$ any vertex of $C$, then $C \setminus \{v\}$ has a perfect matching.
   
   • Use Hall’s theorem to show that $G$ has a perfect matching.

10. For each $k \geq 2$, show that there is a $(2k - 1)$-edge-connected graph that does not have $k$ disjoint spanning trees. (You may begin by showing that for any $q$, a $q$-regular graph with at most $2q$ vertices is $q$-edge-connected.)

11. We claim that any sequence $x_1, x_2, \ldots, x_{rs+1}$ of $rs+1$ real numbers contains a nondecreasing subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$.
   
   (a) Show that the claim follows from Dilworth’s theorem.
   
   (b) Prove the claim directly, without using Dilworth’s or Gallai-Milgram’s theorem. Hint: for each element $x_i$ of the sequence, consider the length $l(x_i)$ of the longest nondecreasing subsequence ending in $x_i$ and the length $l'(x_i)$ of the longest decreasing subsequence ending in $x_i$. 
12. Let $G = (V,E)$ be a directed graph, let $\chi(G)$ be its chromatic number (i.e., $\chi(G)$ is the smallest number of colors needed to color the vertices of $G$ so that no two adjacent vertices have the same color). Show that $G$ contains a directed path with at least $\chi(G)$ vertices. If you can prove this for any directed graph $G$, you get 5 points, if you can only prove it for a graph $G$ that has no directed cycles, you get 3 points.

13. For each of the following decision problems, show that the problem is NP-hard or find a polynomial algorithm. A formula always means a formula in conjunctive normal form. A $k$-formula is a formula whose every clause has $k$ literals. A positive formula is a formula that does not contain any negated literal.

(a) (2-SAT) Input: a 2-formula $F$. Question: does $F$ have a satisfying assignment?

(b) (NAE-SAT) Input: a formula $F$. Question: does $F$ have an assignment such that in each clause at least one literal is satisfied and at least one literal is not satisfied? (Note: a constant like “TRUE” or “FALSE” cannot be used as a literal in a formula. Each literal must be a variable or a negated variable.)

(c) (positive NAE-SAT) like NAE-SAT, but the input is a positive formula.

(d) (positive SAT) like SAT, but the input is a positive formula.

(e) (2-NAE-SAT) like NAE-SAT, but the input is a 2-formula.

(f) (ODD-SAT) Input: a formula $F$. Question: is there an assignment that satisfies an odd number of literals in each clause of $F$?

(g) (Hypergraph bicoloring) Input: a hypergraph $H$. Question: does $H$ have a bicoloring? (See the summary of the recitation of 23. 10. for the necessary definitions.)

14. Show that for each $k$, there is an unsatisfiable $(k, 2^k)$-formula. (Recall that a $(k, 2^k)$-formula is a formula in conjunctive normal form whose clauses have $k$ different literals and each variable appears in at most $2^k$ clauses.)