Fourth recitation session, May 18

We first looked at examples of problems leading to rational generating functions.

Problem 1. An *integer composition of size n with k parts* is an ordered k-tuple $(p_1, \ldots, p_k)$ of positive integers such that $p_1 + \cdots + p_k = n$. Let $a_{n,k}$ be the number of integer compositions of size n with k parts, and define, for a fixed k, the corresponding generating function $A_k(x) = \sum_{n\geq 0} a_{n,k} x^n$. What is $A_1(x)$? How can you express $A_k(x)$ using $A_1(x)$? Let a_n be the number of all integer compositions of size n with any number of parts. Find a formula for the generating function $A(x) = \sum_{n\geq 0} a_n x^n$, and show how to deduce a formula for a_n from the formula for $A(x)$. Also, show how to deduce the formula for a_n by an elementary combinatorial argument.

Problem 2. Let b_n be the number of integer compositions of size n with an even number of parts, and let b'_n be the number of integer compositions of size n with an odd number of parts. Determine, either by a direct combinatorial argument or by generating functions, the asymptotic growth of $|b_n - b'_n|$. Solve the same problem for $|c_n - c'_n|$, with c_n counting integer compositions whose number of parts is a multiple of 3, while c'_n counts those whose number of parts is 1 modulo 3. Is $|c_n - c'_n|$ bounded as $n \to \infty$? Do the same for $|d_n - d'_n|$ where d_n counts compositions whose number of parts is a multiple of 4, and d'_n counts those whose number of parts is 1 mod 4.

We then looked at some theoretical properties of meromorphic functions:

Problem 3. Suppose f is a function with a pole of degree d in a point $z_0 \in \mathbb{C}$. Show that, for a suitable $\varepsilon > 0$ and every $z \in B_{<\varepsilon}(z_0) \setminus \{z_0\}$, the function f admits the following series expansion, where $a_{-d}, a_{-d+1}, \ldots$ are complex coefficients:

$$f(z) = \sum_{n\geq -d} a_n (z-z_0)^n = \frac{a_{-d}}{(z-z_0)^d} + \frac{a_{-d+1}}{(z-z_0)^{d-1}} + \ldots + a_0 + a_1(z-z_0) + \ldots$$

In particular, f can be expressed as a sum of a rational function which is analytic on $\mathbb{C} \setminus \{z_0\}$, and a function that is analytic in z_0.

Problem 4. Suppose that f is a function which is meromorphic on a domain Ω, and which has only finitely many poles in Ω. Show that there is a rational function $r(z)$ and a function $g(z)$ analytic on Ω such that $f(z) = r(z) + g(z)$ for any $z \in \Omega$ which is not a pole of f.

Problem 5. Let f and g be functions analytic on a domain Ω, and such that g is not identically zero on Ω. Show that the function $h(z) = f(z)/g(z)$ is meromorphic on Ω.