Online Chromatic Number is PSPACE-Complete

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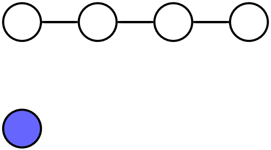
- **Goal:** Minimize no. of colors.
- **Our stronger model:** Algorithm gets a copy of G at the start.

An example



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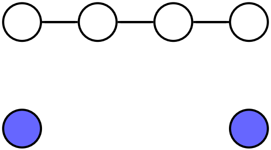

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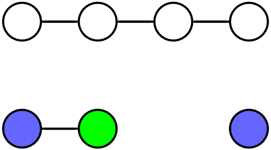

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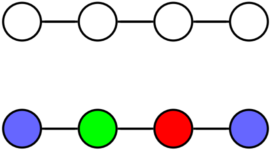

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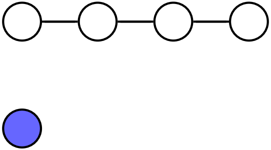

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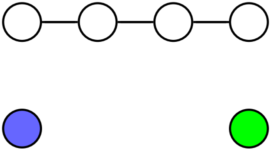

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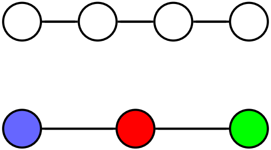

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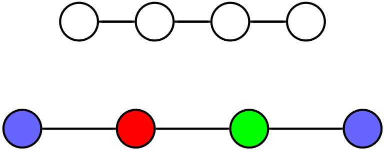

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History

- Online graph coloring first appears in [Bean '76].
- Online chromatic number first appears in [Gyárfás, Lehel '90].
- A copy of a graph at the start – [Halldórsson '96].

Game view

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    - DRAWER (SKETCHER) chooses the next vertex for PAINTER.
    - PAINTER paints a presented vertex with a color
- DRAWER wins when PAINTER uses $k+1$ colors
- Otherwise PAINTER wins
- PAINTER has a winning strategy $\Leftrightarrow \chi^{O}(G) \leq k$

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Complexity and our contribution

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Our contribution: $\chi^O(G) \leq k$ is PSPACE-complete.

Starting step: Precoloring

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Theorem (Kudahl '15)

It is PSPACE-complete to decide $\chi^O(G) \leq k$ given that polynomially many vertices are precolored.

Three steps to the theorem

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A PSPACE-Complete Problem

The problem we reduce to: Q3DNF-SAT.

- **Input:** a fully quantified 3DNF formula.
- **Question:** Is it satisfiable? $\iff$ Is at least one clause satisfiable?

$$\forall x_1 \exists x_2 \forall x_3 \colon (x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3)$$

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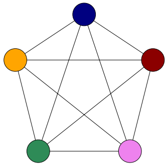
2. PSPACE is closed under complement.
3. Q3DNF-SAT is the complement of Q3CNF-SAT.

Step 1: Precoloring

We precolor a big clique K_{col} with $|V(K_{col})| \approx$ the size of the formula.



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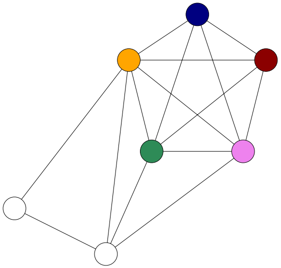

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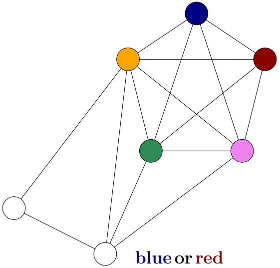

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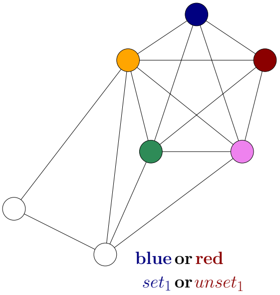
3. Assign meaning to colors.



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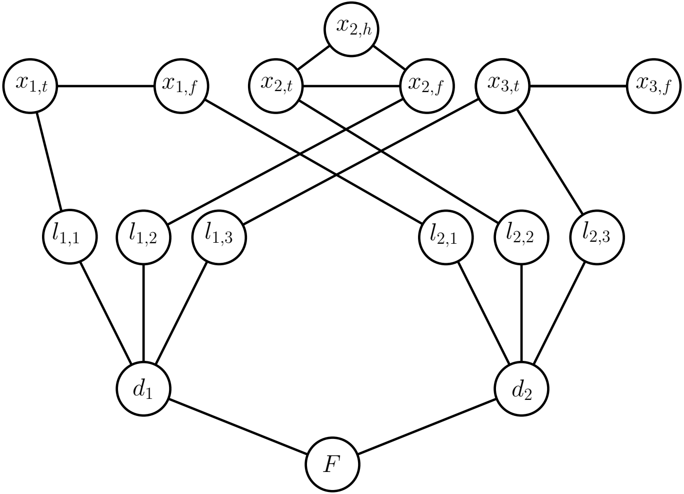
take a polynomially sized K_{col} and add gadgets for each

- variable,
- clause,
- and the entire formula.

Step 1: Gadget sketch

$$\forall x_1 \exists x_2 \forall x_3 : (x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3)$$



Step 1: Variable gadget

$$\forall x_1 \exists x_2 \forall x_3 \colon (x_1 \land \neg x_2 \land x_3) \lor (\neg x_1 \land x_2 \land x_3)$$

1. *Universal quantifier:* two vertices $x_{1,t}$, $x_{1,f}$. Only two allowed colors: set_1, $unset_1$.
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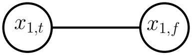

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   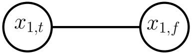

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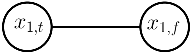

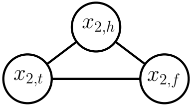

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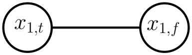

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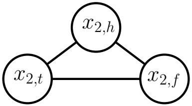

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   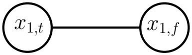

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   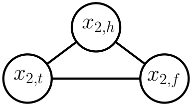

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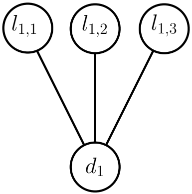
Literal vertex: Two allowed colors, depending on the quantifier. One of the colors always f_a.



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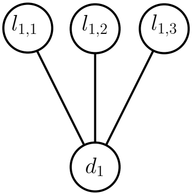

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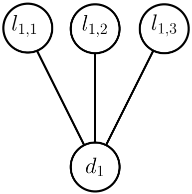

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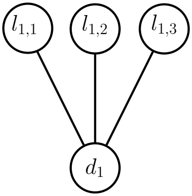

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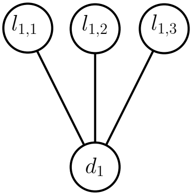

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Final vertex F corresponding to the evaluation of ϕ

- We show: F can be colored $\Leftrightarrow$ ϕ is satisfiable

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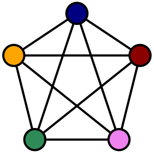

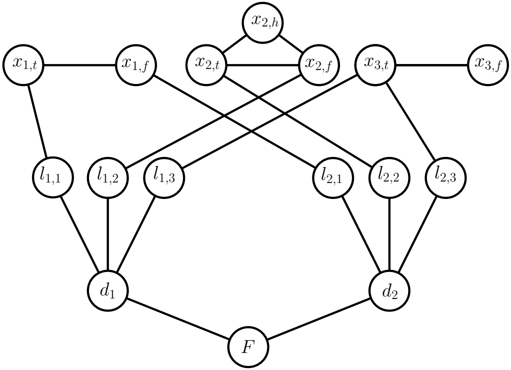

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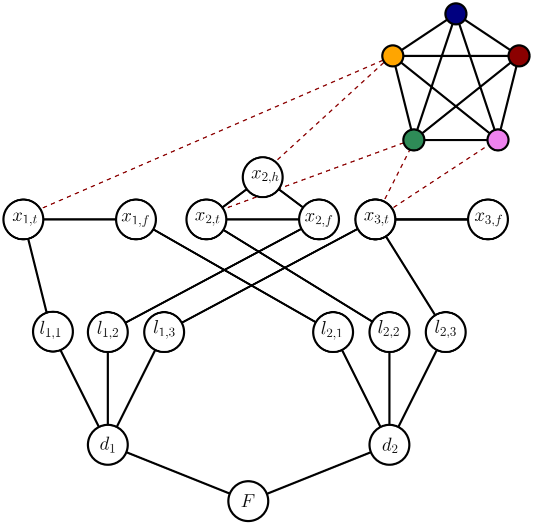

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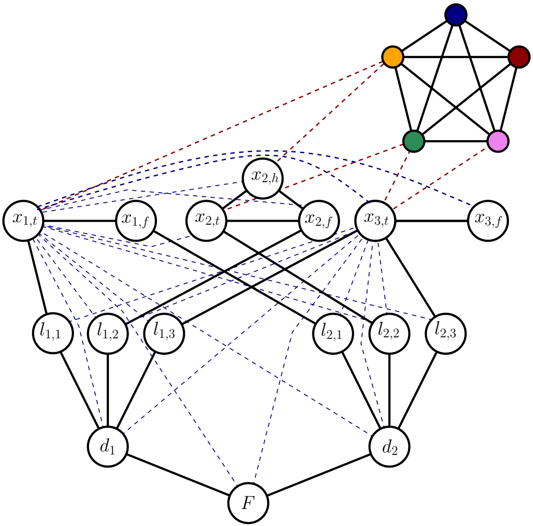

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  - PAINTER can recognize vertices by allowed colors

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- If the formula is not satisfiable, then DRAWER can force PAINTER to use more than k colors
  - DRAWER presents vertices for variables in the right order
- If the formula is satisfiable, then PAINTER can color the graph using k colors
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    - With the exception of vertices of a universally quantified variable

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  - PAINTER can recognize vertices by allowed colors
    - With the exception of vertices of a universally quantified variable
  - Right order of vertices $\Rightarrow$ PAINTER uses the satisfiability
  - Bad order only helps PAINTER

Three steps to the theorem

Theorem

It is PSPACE-complete to decide whether $\chi^O(G) \leq k$.

Step 1:

…with polynomially many precolored vertices (new proof).

Step 2:

…with logarithmically many precolored vertices.

Step 3:

…with no precolored vertices.

Step 2: Sketch

The big clique K_{col} present but uncolored.

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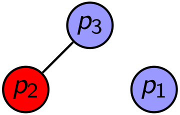
Add one *node* for every "Step 1" vertex.

Single node:



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- Nodes arrive after gadgets: PAINTER saves many colors

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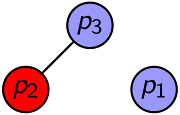

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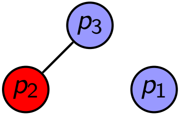

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Only $O(\log n)$ precolored vertices

- to recognize n nodes using binary encoding.

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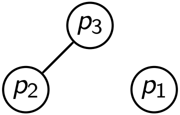

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- The rest of the graph is C-times smaller.

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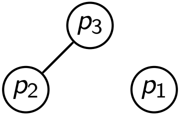

- **Difference:** p_1, p_2, p_3 are now huge cliques.
- The rest of the graph is C-times smaller.

- Either a significant part of the supernode arrives . . .
- . . . or PAINTER can save so many colors the rest can be colored arbitrarily.

Thank you!