Graph polynomials from simple graph sequences

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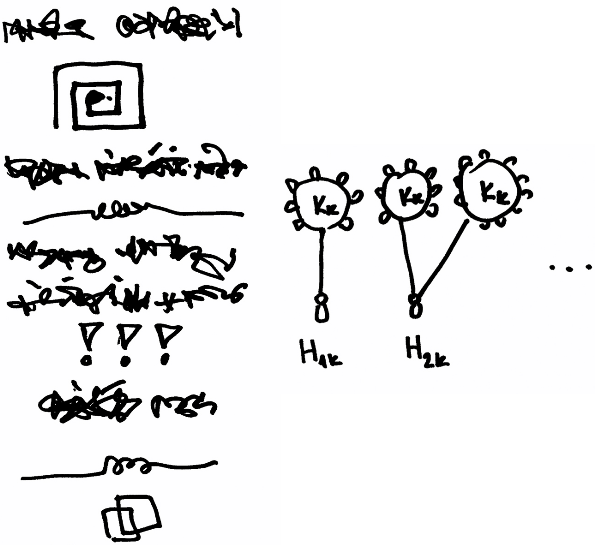
K_k
K_k
K_k
H_{1k}
H_{2k}

Chromatic polynomial

Definition by evaluations at positive integers

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Tutte polynomial

$T(G;x,y)$ universal graph invariant for deletion-contraction of edge e:

$$T(G;x,y) = T(G/e;x,y) + T(G\backslash e;x,y),$$

$$T(G;x,y) = xT(G/\text{bridge};x,y), \quad T(G;x,y) = yT(G\backslash\text{loop};x,y).$$

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For example,

$$P(G;k) = (-1)^{|V(G)|-c(G)}k^{c(G)}T(G;1-k,0).$$

Independence polynomial

Definition by coefficients

$$I(G;x) = \sum_{1\leq j\leq |V(G)|} b_j(G)x^j,$$

$$b_j(G) = \#\{\text{independent subsets of } V(G) \text{ of size } j\}.$$

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$$v \in V(G), \qquad I(G;x) = I(G-v;x) + xI(G-N[v];x)$$

Definition

Graphs G, H.
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$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{uv\in E(G)} a_{f(u),f(v)}$$

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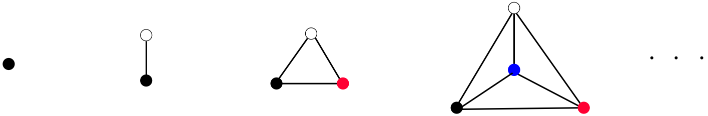
H simple ($a_{s,t} \in \{0,1\}$) or multigraph ($a_{s,t} \in \mathbb{N}$):

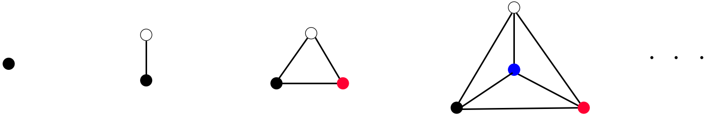
$$\mathrm{hom}(G,H) = \#\{\text{homomorphisms from } G \text{ to } H\}$$
$$= \#\{H\text{-colourings of } G\}$$

Example 1



(K_k)

Example 1



$$(K_k)$$

$$\hom(G, K_k) = P(G; k)$$

chromatic polynomial

Problem

Which sequences (H_k) of graphs are such that, for all graphs G, there is a fixed polynomial $p(G)$ with

$$\hom(G, H_k) = p(G; k)$$

for each $k \in \mathbb{N}$?

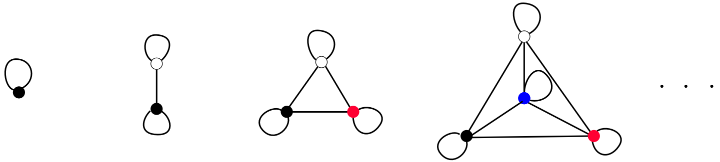
Problem

Which double sequences $(H_{k,\ell})$ of graphs are such that, for all graphs G, there is a fixed bivariate polynomial $p(G)$ with

$$\hom(G, H_{k,\ell}) = p(G; k, \ell)$$

for each $k, \ell \in \mathbb{N}$?

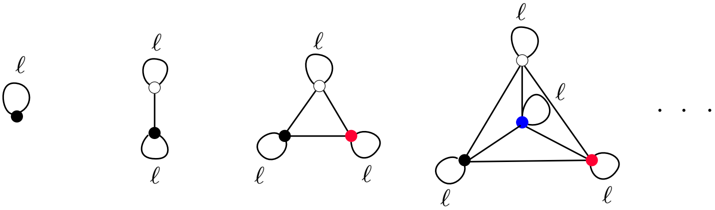
Example 2



$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

Example 3



$$(K_k^\ell)$$

$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

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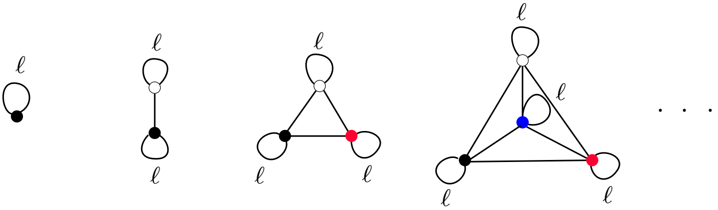

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$$\hom(G, K_k^\ell) = \sum_{f:V(G)\to[k]} \ell^{\#\{uv\in E(G)\,|\,f(u)=f(v)\}}$$

$$= k^{c(G)}(\ell-1)^{r(G)}T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$ (Tutte polynomial)

Example 4

$$(K_1^1 + K_{1,k})$$

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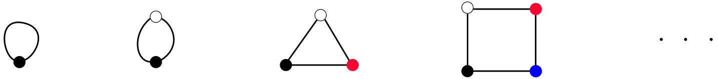
$$\hom(G, K_1^1 + K_{1,k}) = I(G; k)$$

independence polynomial

Non-Example



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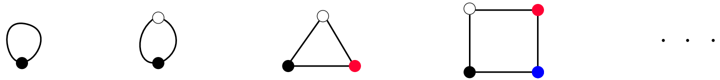

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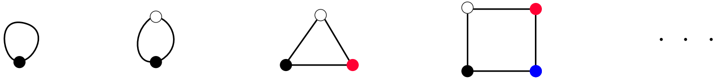

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$$\hom(K_1, C_k) = k,$$

$$\hom(K_2, C_1) = 1, \quad \hom(K_2, C_k) = 2k \text{ when } k \geq 2$$

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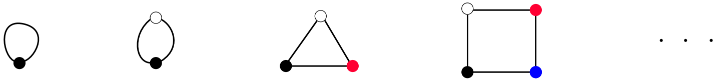

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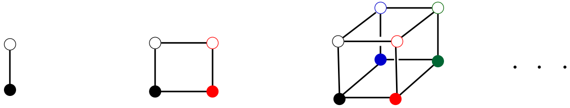
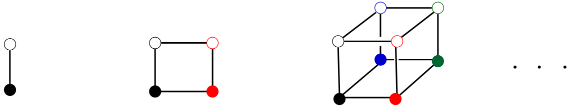
$$\hom(K_3, C_1) = 1, \hom(K_3, C_2) = 0, \hom(K_3, C_3) = 6,$$
$$\hom(K_3, C_k) = 0 \text{ when } k \geq 4$$

Sort-of-Example 5



$(K_2^{\square k}) = (Q_k)$ *(hypercubes)*

Sort-of-Example 5



$$(K_2^{\square k}) = (Q_k) \text{ (hypercubes)}$$

Proposition (Garijo, G., Nešetřil, 2013+)

$\hom(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

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(H_k) is *strongly polynomial* (in k) if $\forall G$ $\exists$ polynomial $p(G)$ such that $\hom(G, H_k) = p(G; k)$ for all $k \in \mathbb{N}$.

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- (K_k^ℓ) is strongly polynomial (in k, ℓ)

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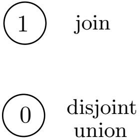
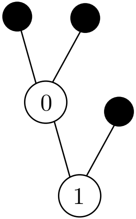
Proposition (de la Harpe & Jaeger 1995)

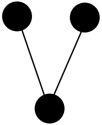
Simple graphs (H_k) *form strongly polynomial sequence* $\iff$ $\forall$*connected* S $\#\{\text{induced subgraphs} \cong S \text{ in } H_k\}$ *polynomial in* k

Cotrees





cotree



...and the
graph it represents

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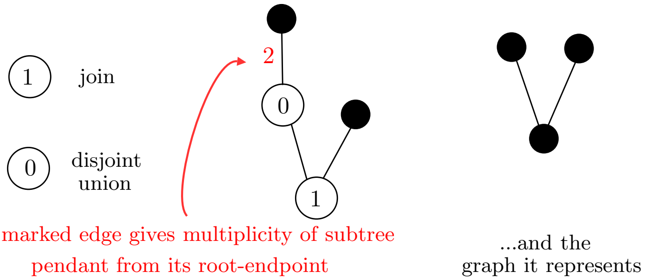
1
join
0
disjoint
union
2
0
1
marked edge gives multiplicity of subtree
pendant from its root-endpoint
...and the
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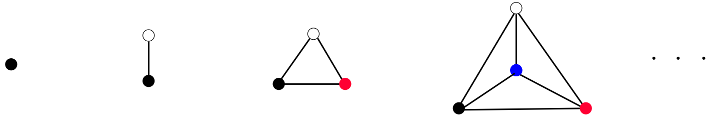
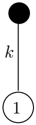
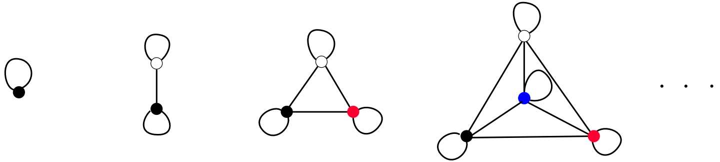


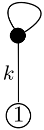
(K_k) — *chromatic polynomial*



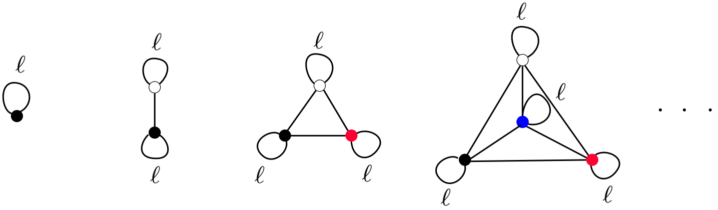
Example 2



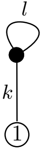
$$(K_k^1)$$



Example 3

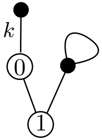


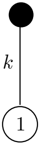
(K_k^ℓ) — *Potts model/ Tutte polynomial*



Example 4

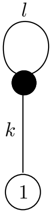
$(K_1^1 + K_{1,k})$ — *Independence polynomial*





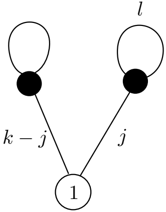
K_k

chromatic



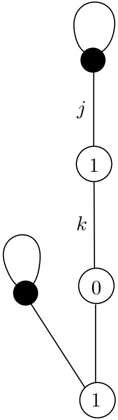
K_k^l

Potts



$K_{k-j}^1 + K_j^l$

Averbouch–Godlin–Makowsky
($l = 0$ is Dohmen–Ponitz–Tittmann)



$K_{1,k}[K_1^1{}_{\text{(centre)}}; K_j^1{}_{\text{(leaves)}}]$

Tittmann–Averbouch–Makowsky

Construction

[Garijo, G., Nešetřil, 2013+] Strongly polynomial sequences in $k, l, \dots$ by representation of graphs by coloured rooted trees (such as cotrees, clique-width parse trees, m-partite cotrees, tree-depth embeddings in closures of rooted trees) with edges marked by polynomials in $k, l, \dots$.

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Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \ldots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

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Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

For every ℓ, D, sequence $(J_{k,\ell,D})$ is strongly polynomial (in k).

However, apart from cocliques and cliques, and the same graphs with a loop on each vertex, the sequence $(J_{k,\ell,D})$ seems not to be generated by our coloured rooted tree construction.

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- $\forall F \quad \exists$ polynomial $q(F) \quad \forall k \in \mathbb{N}: \quad \mathrm{ind}(F, H_k) = q(F; k)$

Satisfaction sets

Quantifier-free formula ϕ with p free variables ($\phi \in \mathrm{QF}_p$) with symbols from relational structure $\mathbf{H}$ with domain $V(\mathbf{H})$.

Satisfaction set $\phi(\mathbf{H}) = \{(v_1, \ldots, v_n) \in V(\mathbf{H})^n : \mathbf{H} \models \phi\}$.

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e.g. for graph structure H (symmetric binary relation $x \sim y$ interpreted as x adjacent to y), and given graph G on n vertices,

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$$|\phi_G(H)| = \hom(G, H).$$

Strongly polynomial sequences of relational structures

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$\forall \phi \in QF \quad \exists$ polynomial $r(\phi) \quad \forall k \in \mathbb{N}: \quad |\phi(\mathbf{H}_k)| = r(\phi; k)$

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Lemma

Equivalently,

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Transitive tournaments $(\vec{T}_k)$ strongly polynomial sequence of digraphs.

Graphical QF interpretation schemes

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Definition (Graphical QF interpretation scheme)

Exponent $p \in \mathbb{N}$, formula $\iota \in \mathrm{QF}_p(\sigma)$ and symmetric formula $\rho \in \mathrm{QF}_{2p}(\sigma)$.
For every σ-structure $\mathbf{A}$, the interpretation $I(\mathbf{A})$ has

$$\text{vertex set} \qquad V = \iota(\mathbf{A}),$$

$$\text{edge set} \qquad E = \{\{\mathbf{u},\mathbf{v}\} \in V \times V : \ \mathbf{A} \models \rho(\mathbf{u},\mathbf{v})\}.$$

Graphical QF interpretation schemes

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- (K_k from $\vec{T}_k$) $p = 1$, $\iota = 1$, $\rho(x, y) = (x < y) \vee (y < x)$ ($x < y$ directed edge in $\vec{T}_k$)
- (C_k from $\vec{T}_k$) $p = 1$, $\iota = 1$, $\rho(x, y) = \rho'(x, y) \vee \rho'(x, y)$,

$$\rho'(x, y) = [x < y \wedge (x < z < y \rightarrow z = x \vee z = y)] \vee \qquad i, i+1$$
$$[\forall z (z < x \vee z = x) \wedge \forall z (y < z \vee y = z)] \qquad k, 1$$

Example (Kneser graphs $J_{k,\ell,\{0\}}$)

- $p = \ell$,

$$\iota(x_1, \ldots, x_\ell) = \bigwedge_{i=1}^{\ell-1} (x_i < x_{i+1})$$

$$\rho(x_1, \ldots, x_\ell, y_1, \ldots, y_\ell) = \bigwedge_{i,j} \neg(x_i = y_j)$$

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- graphs represented (interpreted in) coloured rooted trees.

Graphical QF interpretation schemes

$$I : \text{Relational } \sigma\text{-structures } \mathbf{A} \quad \longrightarrow \quad \text{Graphs } H$$

Lemma

There is

$$\widetilde{I} : \mathrm{QF}(\mathrm{Graphs}) \quad \longmapsto \quad \mathrm{QF}(\sigma\textit{-structures})$$

such that

$$\phi(I(\mathbf{A})) = \widetilde{I}(\phi)(\mathbf{A})$$

In particular, $(\mathbf{A}_k)$ *strongly polynomial* $\Rightarrow$ $(H_k) = (I(\mathbf{A}_k))$ *strongly polynomial.*

From graphs to graphs

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- Cartesian product and other more complicated graph products are special kinds of such interpretation schemes too.

Example

- (Cartesian product of graphs G_1 and G_2)

$$\mathbf{A} = G_1 \sqcup G_2$$

$$U_i(v) \quad \Leftrightarrow \quad v \in V(G_i),$$
$$R_i(u,v) \quad \Leftrightarrow \quad uv \in E(G_i) \quad (i = 1, 2)$$

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Interpretation scheme I of exponent $p = 2$ defined on (U_1, U_2, R_1, R_2)-relational structures $\mathbf{A}$ by

$$\iota(x_1, x_2) : U_1(x_1) \wedge U_2(x_2)$$

$$\rho(x_1, x_2, y_1, y_2) : [R_1(x_1, y_1) \wedge (x_2 = y_2)] \vee [(x_1 = y_1) \wedge R_2(x_2, y_2)]$$

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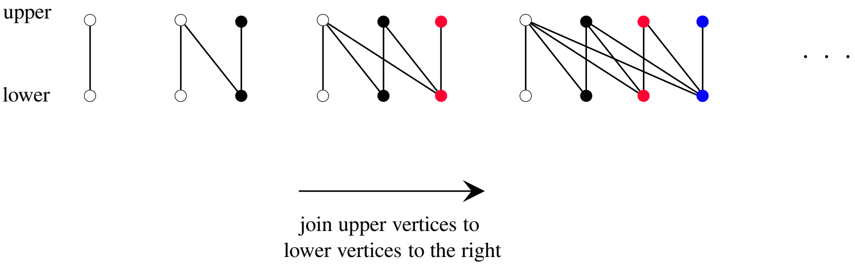
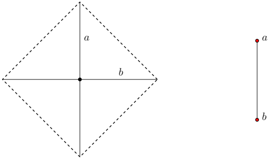
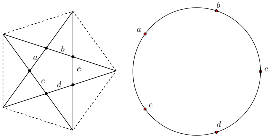
- Half-graphs are QF interpretations of $\vec{T}_k$ together with $\vec{T}_2$ and two unary relations to specify "upper" and "lower" vertices, and so form a strongly polynomial sequence.



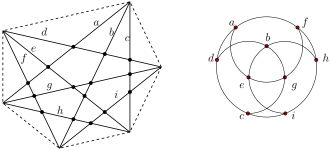
- Intersection graphs of chords of a k-gon form a strongly polynomial sequence



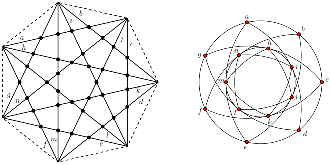
(a) Square



(b) Pentagon



(c) Hexagon



(d) Heptagon

Conjecture

All strongly polynomial sequences of graphs (H_k) such that $H_k \subseteq_{\mathrm{ind}} H_{k+1}$ can be obtained by QF interpretation of a "basic sequence" (finite disjoint union of transitive tournaments of size polynomial in k with unary relations).

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Theorem (G., Nešetřil, Ossona de Mendez , 2014+)

A sequence (H_k) of graphs of uniformly bounded degree is a strongly polynomial sequence if and only if it is a QF-interpretation of a basic sequence.

- When is $\mathrm{hom}(G, \mathrm{Cayley}(A_k, B_k))$ a fixed polynomial (dependent on G) in $|A_k|, |B_k|$, where $B_k = -B_k \subseteq A_k$?

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- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial and independence polynomial?
- Which strongly polynomial sequences of graphs give matroid invariants when suitably scaled like the chromatic polynomial? e.g. $\binom{k}{\ell}^{-c(G)}\hom(G, J_{k,\ell,D})$ [de la Harpe & Jaeger, 1995]

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Theorem (G., Nešetřil, Ossona de Mendez , 2014+)

If (H_k) is strongly polynomial then there are only finitely many terms that belong to a quasi-random sequence of graphs.

Beyond polynomials? Rational generating functions

- For strongly polynomial sequence (H_k),

$$\sum_k \hom(G, H_k) t^k = \frac{P_G(t)}{(1-t)^{|V(G)|+1}}$$

with polynomial $P_G(t)$ of degree at most $|V(G)|$.

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- For eventually polynomial sequence (H_k) such as (C_k),

$$\sum_k \hom(G, H_k) t^k = \frac{P_G(t)}{(1-t)^{|V(G)|+1}}$$

  with polynomial $P_G(t)$.

Beyond polynomials? Rational generating functions

- For quasipolynomial sequence of Turán graphs $(T_{k,r})$

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- For sequence of hypercubes (Q_k),

$$\sum_k \hom(G, Q_k)t^k = \frac{P_G(t)}{Q(t)^{|V(G)|+1}}$$

with polynomial $P_G(t)$ of degree at most $|V(G)|$ and polynomial $Q(t)$ with zeros powers of 2.

Beyond polynomials? Algebraic generating functions

- For sequence of odd graphs $O_k = J_{2k-1,k-1,\{0\}}$, is

$$\sum_k \mathrm{hom}(G, O_k)t^k$$

algebraic? (e.g. it is $\frac{1}{2}(1-4t)^{-\frac{1}{2}}$ when $G = K_1$).

Three papers

- P. de la Harpe and F. Jaeger, Chromatic invariants for finite graphs: theme and polynomial variations, *Lin. Algebra Appl.* **226–228** (1995), 687–722

  Defining graphs invariants from counting graph homomorphisms. Examples. Basic constructions.

- D. Garijo, A. Goodall, J. Nešetřil, Polynomial graph invariants from homomorphism numbers. 40pp. arXiv: 1308.3999 [math.CO] Further examples. New construction using coloured rooted tree representations of graphs.

- A. Goodall, J. Nešetřil, P. Ossona de Mendez, Strongly polynomial sequences as interpretation of trivial structures. 21pp. arXiv:1405.2449 [math.CO] General relational structures: counting satisfying assignments for quantifier-free formulas. Building new polynomial invariants by interpretation of "trivial" sequences of marked tournaments.