Graph invariants, homomorphisms, and the Tutte polynomial

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3 The Tutte polynomial

Let A be an additive Abelian group of order k. The number of nowhere-zero A-flows of a planar graph G, $F(G;k)$, is under duality equal to the number of nowhere-zero A-tensions of G^*, $F^*(G;k)$. (This was discussed in the chapter on flows and tensions, Cor. 15 and Prop. 17.) For a planar graph we have $F^*(G;k) = F(G^*;k)$. By the correspondence between nowhere-zero tensions and proper vertex colourings we have $P(G;k) = k^{c(G)}F^*(G;k)$.

Upon switching deletion and contraction, the deletion-contraction recurrence satisfied by $F(G;k)$ is that satisfied by $F^*(G;k)$. We shall see further examples of how deletion and contraction are dual operations. We also have seen (Theorem 18 in the chapter on the chromatic polynomial) that $|P(G;-1)| = |F^*(G;-1)|$ is equal to the number of acyclic orientations of G, and dually $|F(G;-1)|$ is equal to the number of totally cyclic orientations of G. Indeed, a totally cyclic orientation is an orientation containing no directed cocircuit, which is dual to an acyclic orientation (containing no directed circuit).

Not everything that can be computed recursively by deletion-contraction can be found as an evaluation of the chromatic or flow polynomial, however. The number of spanning forests of a connected graph G (subsets $F \subseteq E$ that contain no circuit of G), and the number of spanning subgraphs of G with $c(G)$ components (subsets F whose complement $E \setminus F$ contains no cocircuit of G), are both quantities satisfying a deletion-contraction rule (see end of the chapter on the chromatic polynomial). But it is clear that neither of these graph invariants can be calculated from the chromatic and flow polynomials alone (why?).

In this chapter we shall see how the number of spanning forests and a wide range of other graph invariants can also be obtained by evaluations of a bivariate polynomial that includes the chromatic polynomial and flow polynomial as specializations. Moreover this bivariate polynomial is "universal" for deletion-contraction invariants.

3.1 Defining the Tutte polynomial

Recall that the rank of G is defined by $r(G) = |V(G)| - c(G)$ and is the dimension of the cutset space of G ($\mathbb{Z}_2$-tensions of G). The *nullity* of G is defined by $n(G) = |E(G)| - r(G)$ and is the dimension of the cycle space of G ($\mathbb{Z}_2$-flows of G).

It will be convenient to call an edge *ordinary* when it is neither a bridge nor a loop.

Consider the following recursive definition of a graph invariant $T(G;x,y)$ in two independent variables x and y. If G has no edges then $T(G;x,y) = 1$, otherwise, for any $e \in E(G)$,

$$T(G;x,y) = \begin{cases} T(G/e;x,y) + T(G\backslash e;x,y) & e \text{ ordinary,} \\ xT(G/e;x,y) & e \text{ a bridge,} \\ yT(G\backslash e;x,y) & e \text{ a loop.} \end{cases} \tag{1}$$

By induction this defines a bivariate polynomial $T(G;x,y)$, called the *Tutte polynomial* of G, all of whose coefficients are non-negative integers. See Figure 1 for a couple of small examples illustrating how the Tutte polynomial can be computed recursively, in a similar way to the chromatic polynomial in the previous chapter, except without the complication of needing to keep track of signs.

It is not immediately clear that it does not matter which order the edges are chosen to calculate $T(G; x, y)$ recursively using (1). (For the chromatic polynomial the situation was different: we had a well-defined graph polynomial and we proved it satisfied a deletion-contraction recurrence.)

Proposition 1. *If e_1 and e_2 are distinct edges of G then the outcome of first applying the recurrence* (1) *with edge e_1 and then with edge e_2 is the same as with the reverse order, when first taking e_2 and then e_1.*

Proof. (Sketch) First observe that if e_1 and e_2 are parallel then the statement is clearly true (swapping e_1 and e_2 is an automorphism of G). When e_1 and e_2 are not parallel, the type of edge e_2 in G (whether it is a bridge, loop, or ordinary) is preserved in G/e_1 and in $G\backslash e_1$. For each of the possible combinations of edge types for e_1 and e_2, one verifies that swapping the order of e_1 and e_2 gives the same outcome in the two-level computation tree going from G to G with edges e_1 and e_2 deleted or contracted. For example, if both edges are ordinary then the truth of the statement amounts to the fact that $G/e_1\backslash e_2 \cong G\backslash e_2/e_1$ and similarly for the other three combinations of deletion and contraction. ☐

The recurrence (1) can be restated as follows. If G consists of k bridges and ℓ loops then $T(G; x, y) = x^k y^\ell$, otherwise $T(G; x, y) = T(G/e; x, y) + T(G\backslash e; x, y)$ for an ordinary edge e of G.

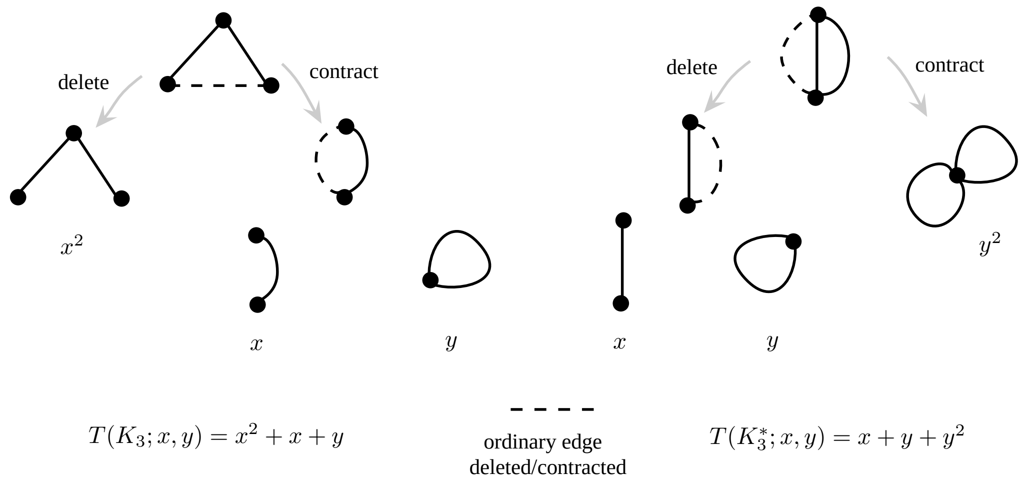

Figure 1: Using deletion-contraction to compute the Tutte polynomial of K_3 and its dual K_3^*.

> **Question 1** For any given $n \geq 1$, calculate the Tutte polynomial of the cycle C_n.

A graph G is 2-connected if and only if has no cut-vertex. A graph is 2-connected if and only if its cycle matroid is connected (not the direct sum of two smaller matroids). A loop on a single vertex (C_1) and a single bridge (K_2) are both 2-connected. For the case of many loops on a single vertex (where one might still consider the vertex not to be a cut-vertex) we refer to the cycle matroid, which is the direct sum of its constituent loops: so this graph is not 2-connected when there is more than one loop.

A *block* of G is a maximal 2-connected induced subgraph of G. If G is not 2-connected then it can be written in the form $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| \leq 1$. The intersection graph of the blocks of a loopless connected graph is a tree. In particular, if G is loopless and connected and has at least two blocks then there are at least two *endblocks* of G which are blocks containing only one cut-vertex of G.

Proposition 2. *The Tutte polynomial of G is multiplicative over the connected components of G and over the blocks of G: if $G = G_1 \cup G_2$ where G_1 and G_2 share at most one vertex then $T(G_1 \cup G_2; x, y) = T(G_1; x, y)T(G_2; x, y)$.*

Proof. The statement is true when each edge is either a bridge or a loop, since in this case $T(G;x,y) = x^k y^\ell$, where k is the number of bridges and ℓ the number of loops. We argue by induction on the number of ordinary edges of G. Let $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = 1$. The endpoints of any edge e must belong to the same block of G; if e is a bridge or loop then it forms its own block. Suppose $G = G_1 \cup G_2$ where G_1 is a block of G containing an ordinary edge e. Deleting or contracting e can only decrease the number of ordinary edges of G and since e is ordinary we have, writing $T(G;x,y) = T(G)$,

$$\begin{aligned}
T(G) &= T(G/e) + T(G\backslash e) \\
&= T(G_1/e \cup G_2) + T(G_1\backslash e \cup G_2) \\
&= [T(G_1/e) + T(G_1\backslash e)]T(G_2) \\
&= T(G_1)T(G_2),
\end{aligned}$$

where to obtain the third line we applied the inductive hypothesis. □

The converse to Proposition 2 also holds, although its proof is bit more involved:

Theorem 3. [39] *If G is a loopless 2-connected graph then $T(G;x,y)$ is irreducible in $\mathbb{Z}[x,y]$.*

The factors of the Tutte polynomial of G therefore correspond precisely to the blocks of G.

An *open ear decomposition* of a graph $G = (V,E)$ is a partition of the edges of G into a sequence of simple paths $P_0, P_1, \ldots, P_s$ such that P_0 is a single edge, each endpoint of P_i, $1 \le i \le s$, is contained in some P_j, $j < i$, and the internal vertices of P_i are not vertices of any other P_j, $j < i$. The P_i are called the *ears* of the decomposition. A graph has an open ear decomposition if and only if it is 2-connected [55]. The length s of an open ear decomposition of G equals its nullity, i. e., $s = |E| - |V| + 1$.

Question 2 The last ear P in an open ear decomposition of G is an induced path. Let $G\backslash P$ denote the minor of G obtained by deleting all the edges in P, and G/P the minor obtained by successively contracting all the edges in P (in any order).

(i) Prove that if P has k edges then

$$T(G;x,y) = (x^{k-1} + \cdots + x + 1)T(G\backslash P;x,y) + T(G/P;x,y).$$

(When $k = 1$ this is the defining deletion-contraction formula for an ordinary edge.)

(ii) Calculate the Tutte polynomial of the graph formed by identifying the endpoints of each of three paths on a, b and c vertices (a *theta graph*). In particular, write down the polynomials $T(K_{2,3};x,y)$ and $T(K_4^-;x,y)$, where K_4^- is K_4 minus an edge.

(iii) Calculate $T(K_4;x,y)$.

Here are some basic properties of the coefficients of $T(G;x,y)$:

Proposition 4. *For a graph G with Tutte polynomial $T(G;x,y) = \sum t_{i,j}(G)x^i y^j$,*

(i) $t_{0,0}(G) = 0$ if $|E(G)| > 0$;

(ii) if G has no loops then $t_{1,0}(G) \neq 0$ if and only if G is 2-connected;

(iii) x^k divides $T(G;x,y)$ if and only if G has at least k bridges, and y^ℓ divides $T(G;x,y)$ if and only if G has at least ℓ loops;

(iv) given G has k bridges and ℓ loops, if $i \ge r(G)$ or $j \ge n(G)$ then $t_{i,j}(G) = 0$ except when $i = r(G)$ and $j = \ell$, or $i = k$ and $j = n(G)$, where we have $t_{r(G),\ell}(G) = 1 = t_{k,n(G)}(G)$.

Proof. For (ii), we use the property that if G is 2-connected, then at least one of G/e and $G\backslash e$ is also 2-connected. A basis for induction is that $T(K_2; x, y) = x$. Given a loopless graph G, if e is not parallel to another edge then both G/e and $G\backslash e$ have no loops, and the equation $t_{1,0}(G) = t_{1,0}(G/e) + t_{1,0}(G\backslash e)$ provides the inductive step. If e is parallel to another edge then G/e has a loop and $t_{1,0}(G) = t_{1,0}(G\backslash e)$; by deleting all but one edge in a parallel class we can thus assume G is simple. For the converse, if G is not 2-connected then by Proposition 2 its Tutte polynomial is the product of at least two polynomial factors, each corresponding to a block of G; by what we have just proved $t_{1,0}(B) = 1$ for each such block B, and this implies $t_{1,0}(G) = 0$.

For (iv), we shall use induction on the number of ordinary edges to prove that $t_{i,j}(G) = 0$ when $i \geq r(G)$ or $j \geq n(G)$, except for $t_{r(G),\ell}(G) = 1 = t_{k,n(G)}(G)$. The base case is when G has no ordinary edges, consisting of k bridges and ℓ loops. Here $r(G) = k$ and $n(G) = \ell$, and $t_{k,\ell}(G) = 1$, while $t_{i,j}(G) = 0$ for all other values of i, j. Hence the statement is true in this case.

Consider the recurrence formula $t_{i,j}(G) = t_{i,j}(G/e) + t_{i,j}(G\backslash e)$ for an ordinary edge e. We have by inductive hypothesis that $t_{i,j}(G/e) = 0$ for $i \geq r(G/e) = r(G) - 1$ except $t_{r(G)-1,\ell}(G/e) = 1$, and for $j \geq n(G/e) = n(G)$ except $t_{k,n(G)}(G/e) = 1$. This gives $t_{i,j}(G) = 0$ for $j \geq n(G)$ except $t_{k,n(G)}(G) = 1$.

Also $t_{i,j}(G\backslash e) = 0$ for $i \geq r(G\backslash e) = r(G)$ except $t_{r(G),\ell}(G\backslash e) = 1$, and for $j \geq n(G\backslash e) = n(G) - 1$ except $t_{k,n(G)-1}(G/e) = 1$. This gives $t_{r(G),\ell}(G) = 0$ for $i \geq r(G)$ except $t_{r(G),\ell}(G) = 1$. □

If G is a bridgeless loopless 2-connected graph and H is a minor of G having at least one edge then $t_{i,j}(H) \leq t_{i,j}(G)$ (proved by Brylawski [18, Corollary 6.9] in the more general context of matroids). When G is not 2-connected and H is a minor of G, the coefficients of $T(H; x, y)$ are not necessarily dominated by those of $T(G; x, y)$: for an example, take G to be a tree on at least two edges and H any proper minor.

3.2 Evaluations of the Tutte polynomial

We have met quite a few examples of graph invariants that satisfy a deletion-contraction recurrence, such as the chromatic polynomial, the flow polynomial, and the number of acyclic orientations. For the number of acyclic orientations, which we denoted by $Q(G)$ in the proof of Theorem 18 in the chapter on the chromatic polynomial, we derived the recurrence $Q(G) = Q(G\backslash e) + Q(G/e)$. Together with the fact that $Q(K_2) = 2$ and $Q(C_1) = 0$ this implies that in fact the number of acyclic orientations is equal to $T(G; 2, 0)$, which satisfies precisely the same recurrence and boundary conditions on loops and bridges.

Question 3
The number of spanning trees, spanning forests and connected spanning subgraphs are each equal to an evaluation of the Tutte polynomial. At which points?

How about the chromatic polynomial and flow polynomial? We are momentarily halted in our stride by the fact that the recurrence $P(G; z) = P(G\backslash e; z) - P(G/e; z)$ involves a subtraction, which doees not feature in the deletion-contraction recurrence for the Tutte polynomial. However, "momentarily" is the operative word. The following theorem describes the necessary ingredients for other evaluations of the Tutte polynomial. It turns out the Tutte polynomial is all-embracing of graph invariants satisfying deletion-contraction recurrences of the form satisfied by the chromatic polynomial.

Theorem 5. "Recipe Theorem" *Let $\mathcal{G}$ be a minor-closed class of graphs. There is a unique graph invariant $f : \mathcal{G} \to \mathbb{Z}[x, y, \alpha, \beta, \gamma]$ such that $f(\overline{K}_n) = \gamma^n$ for $n = 1, 2, \ldots$, and for every edge $e \in E$*

$$f(G) = \begin{cases} \alpha f(G/e) + \beta f(G\backslash e) & e \text{ not a bridge or loop,} \\ x f(G/e) & e \text{ a bridge,} \\ y f(G\backslash e) & e \text{ a loop.} \end{cases} \tag{2}$$

The graph invariant f is equal to the following specialization of the Tutte polynomial:

$$f(G) = \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \tag{3}$$

NOTE. (i) If instead of contracting a bridge we require that $f(G) = xf(G\backslash e)$ when e is a bridge, the Tutte polynomial is evaluated at the point $(\gamma x/\alpha, y/\beta)$ instead of $(x/\alpha, y/\beta)$. In particular, when $\gamma = 1$ it does not matter whether bridges are deleted or contracted.

(ii) If either α or β is zero then we interpret (3) as the result of substituting values of the parameters after expanding the expression on the right-hand side as a polynomial in $\mathbb{Z}[\alpha, \beta, \gamma, x, y]$. Given a graph G with k bridges and ℓ loops, using Proposition 4 (iv) we see that if $\alpha = 0$ then $f(G) = \gamma^{c(G)}\beta^{n(G)-\ell}x^{r(G)}y^{\ell}$, and if $\beta = 0$ then $f(G) = \gamma^{c(G)}\alpha^{r(G)-k}x^k y^{n(G)}$. If both α and β are zero then $f(G) = 0$ if G has an ordinary edge, while $f(G) = \gamma^{c(G)}x^k y^{\ell}$ if $E(G)$ consists of just k bridges and ℓ loops.

Proof. Uniqueness of $f(G)$ follows by induction on the number of edges and application of the recurrence (2).

Formula (3) is certainly true for cocliques $\overline{K}_n$. If G consists just of k bridges and ℓ loops and has c connected components, then $f(G) = \gamma^c x^k y^{\ell}$ and since $r(G) = k$ and $n(G) = \ell$ we have $T(G; \frac{x}{\alpha}, \frac{y}{\beta}) = \left(\frac{x}{\alpha}\right)^k \left(\frac{y}{\beta}\right)^{\ell}$, so (3) is satisfied. Let e be an ordinary edge, and note that $c(G) = c(G/e) = c(G\backslash e)$, so that $r(G/e) = r(G) - 1$, $r(G \setminus e) = r(G)$ and $n(G/e) = n(G)$, $n(G\backslash e) = n(G) - 1$. By induction on the number of ordinary edges,

$$\begin{aligned}
f(G) &= \alpha f(G/e) + \beta f(G\backslash e) \\
&= \alpha \cdot \gamma^{c(G)}\alpha^{r(G)-1}\beta^{n(G)}T(G/e; \frac{x}{\alpha}, \frac{y}{\beta}) + \beta \cdot \gamma^{c(G)}\alpha^{r(G)}\beta^{n(G)-1}T(G\backslash e; \frac{x}{\alpha}, \frac{y}{\beta}) \\
&= \gamma^{c(G)}\alpha^{r(G)}\beta^{n(G)}T(G; \frac{x}{\alpha}, \frac{y}{\beta}).
\end{aligned}$$

□

A graph invariant satisfying the recurrence (2) is called a *generalized Tutte–Grothendieck invariant*, or TG-invariant for short [19]. (Tutte-Grothendieck rings were introduced by Brylaskwi [17] in an early paper on matroids (or "pregeometries") as a generalization of Tutte's ring in graph theory [47] and using a 'construction reminiscent of constructions recently used with great success in the field of algebraic geometry by A. Grothendieck.') A TG-invariant is multiplicative over disjoint unions, and if G_1 and G_2 share just one vertex then $f(G_1 \cup G_2) = f(G_1)f(G_2)/\gamma$. (The archetypal example is the chromatic polynomial.) See [19] for TG-invariants in graph theory and matroid theory more generally.

Question 4
Suppose the graph invariant $f(G)$ satisfies the recurrence (2). Show that the graph invariant

$$\left(\frac{x-\alpha}{\beta\gamma}\right)^{c(G)} \left(\frac{y-\beta}{\alpha}\right)^{|V(G)|} \delta^{|E(G)|} f(G),$$

where δ is an arbitrary constant, satisfies the recurrence $f(G) = (y-\beta)f(G/e)+\beta f(G\backslash e)$ independently of whether e is a bridge, loop, or ordinary. (For example, the chromatic polynomial is an example of such an invariant, its recurrence $P(G; z) = P(G\backslash e; z) - P(G/e; z)$ holding for any edge e.)

An example we have already seen for the above question is when $f(G)$ is the number of acyclic orientations of G. This is a TG-invariant with $\alpha = \beta = \gamma = 1$ and $x = 2, y = 0$, satisfying $f(G) = f(G/e) + f(G\backslash e)$ when e is not a loop. The invariant $(-1)^{|V(G)|}f(G)$ satisfies $f(G) = f(G \setminus e) - f(G/e)$ for all edges e (as we know from Theorem 18 in the chapter on the chromatic polynomial, it is equal to $P(G; -1)$).

Proposition 6. *The* monochrome polynomial*,*

$$B(G; k, y) = \sum_{f:V(G)\to[k]} y^{\#\{uv\in E(G):f(u)=f(v)\}},$$

is the following specialization of the Tutte polynomial:

$$B(G; k, y) = k^{c(G)}(y-1)^{r(G)}T(G; \frac{y-1+k}{y-1}, y).$$

The chromatic polynomial is given by

$$P(G; z) = (-1)^{r(G)} z^{c(G)} T(G; 1 - z, 0).$$

Proof. Proposition 17 in the chapter on the chromatic polynomial gives the recurrence formula

$$B(G; k, y) = (y - 1)B(G/e; k, y) + B(G\backslash e; k, y), \tag{4}$$

valid for all edges e.

For the chromatic polynomial we have $P(G; z) = (z-1)P(G/e; z)$ when e is a bridge, for we have $P(G\backslash e; z) = zP(G/e; z)$. A direct argument for $P(G \setminus e; k) = kP(G/e; k)$ when $e = uv$ is a bridge is as follows. Suppose $G\backslash e = G_1 \cup G_2$ with $u \in V(G_1)$ and $v \in V(G_2)$. Then G/e is obtained from $G_1 \cup G_2$ by identifying the vertices u and v to make a cut-vertex w. Given a fixed colour $\ell \in [k]$, there are $P(G_1; k)/k$ proper colourings $f_1 : V(G_1) \to [k]$ of G_1 with $f_1(w) = \ell$, and $P(G_2; k)/k$ proper colourings $f_2 : V(G_2) \to [k]$ of G_2 with $f_2(w) = \ell$. Since there are no edges between G_1 and G_2, there are $P(G_1; k)P(G_2; k)/k^2$ proper colourings of G/e with $f(w) = \ell$. This number is independent of ℓ, so there are $P(G_1; k)P(G_2; k)/k$ proper colourings of G/e. On the other hand, there are $P(G_1; k)P(G_2; k)$ proper colourings of $G\backslash e$. Hence $P(G\backslash e; k) = kP(G/e; k)$ when e is a bridge of G.

For the monochrome polynomial, when e is a bridge we have $B(G\backslash e; k, y) = kB(G/e; k, y)$, by a similar argument to the chromatic polynomial, by conditioning on the colour of the cut-vertex w of G/e obtained by identifying the endpoints of e. Instead of proper colourings, consider colourings with exactly m_1 monochrome edges in G_1 and exactly m_2 monochrome edges in G_2. Then the number of such colourings for $G\backslash e$ (the disjoint union of G_1 and G_2) is k times the number for G/e (the gluing of G_1 and G_2 at a vertex). Collecting together all colourings for which $m_1 + m_2 = m$, this implies that the coefficient of y^m in $B(G\backslash e; k, y)$ is equal to k times the corresponding coefficient in $B(G/e; k, y)$. Since this holds for each m, it follows that $B(G\backslash e; k, y) = kB(G/e; k, y)$ when e is a bridge, and so $B(G; k, y) = (y - 1 + k)B(G/e)$ by the recurrence formula (4). When e is a loop $B(G; k, y) = yB(G\backslash e; k, y)$ since a loop is always monochromatic (or by looking at the recurrence formula (4) with $G/e \cong G\backslash e$ when e is a loop).

The result now follows by Theorem 5. □

Let's return to acyclic orientations of G, which we have seen are counted by $T(G; 2, 0) = (-1)^{|V(G)|}P(G; -1)$, for a surprising result concerning them is waiting in the wings.

Question 5

(i) Show that an acyclic orientation of G has at least one source (all edges outgoing) and at least one sink (all edges incoming).

(ii) What is the dual statement for totally cyclic orientations? Prove it.

Theorem 7. [Greene and Zaslavsky [26]] *Suppose $G = (V, E)$ is a connected graph and $u \in V$. Then the number of acyclic orientations of G with unique source at u is equal to $T(G; 1, 0)$. In particular, this number is independent of the choice of u.*

Note when G is connected $T(G; 1, 0) = P'(G; 0)$, the coefficient of z in $P(G; z)$, so Theorem 7 gives another graph invariant that can be calculated using the chromatic polynomial of G alone.

Proof. Fix a vertex u of G and let $Q_u(G)$ denote the number of acyclic orientations with a unique source at u.

Suppose G is connected and with at least one edge. Choose an edge $e = uv$ with one endpoint the source vertex u. (Since G is connected there has to be at least one edge incident with u.)

If e is the only edge of G, then $Q_u(G) = 1$ when e is a bridge, and $Q_u(G) = 0$ when e is a loop. Suppose there are other edges.

If e is a loop then $Q_u(G) = 0$.

If e is a bridge then $Q_u(G) = Q_u(G/e)$. For consider an acyclic orientation $\mathcal{O}$ of G with unique source u. Then in the component of $G\backslash e$ containing v, the only source of $\mathcal{O}$ restricted to this component has to be v, otherwise there would be a source other than u in $\mathcal{O}$. Therefore, acyclic orientations of G with unique source at u are in one-to-one correspondence with acyclic orientations of G/e with unique source at u (which in G/e has been identified with the vertex v).

If e is ordinary then partition acyclic orientations with u as a unique source into two sets: those for which uv is the only edge directed into v (so deleting uv does not give an acyclic orientation of $G\backslash e$ with a unique source) and those for which uv is not the only edge directed into v (here deleting uv gives an acyclic orientation of $G\backslash e$ with unique source at u). The first set is in one-to-one correspondence with acyclic orientations of G/e with unique source at u (in G/e vertex v is identified with vertex u), while the second set is in one-to-one correspondence with acyclic orientations of $G\backslash e$ with unique source at u. Hence when e is ordinary we have $Q_u(G) = Q_u(G/e) + Q_u(G\backslash e)$.

By Proposition 5 it follows that $Q_u(G) = T(G; 1, 0)$. □

Question 6
Formulate and prove the dual version of Theorem 7.

3.3 Subgraph expansion

As we have already alluded to, a way to circumvent the need to prove that the deletion-contraction recurrence (1) gives a well-defined polynomial $T(G; x, y)$ for a graph G, independent of the order in which the edges are chosen, is to exhibit such a polynomial that can be verified to satisfy the recurrence. We begin this section by doing precisely this. First a recap of some notation.

Let $G = (V, E)$ be a graph and $A \subseteq E$. Identify A with the spanning subgraph $G_A = (V, A)$. The rank of A is defined by $r_G(A) = |V(G)| - c(G_A)$ (this is the matroid rank function on the cycle matroid of G). The nullity of A is defined by $n_G(A) = |A| - r_G(A)$. Thus $r_G(E) = r(G)$ and $n_G(E) = n(G)$ in the notation already introduced for the rank and nullity of the graph G. When context makes it clear what graph G is, we drop the subscript and write $r(A)$ for $r_G(A)$ and $n(A)$ for $n_G(A)$.

It is easy to see that $0 \le r(A) \le |A|$ with $r(A) = 0$ if and only if A is empty or a set of loops, and $r(A) = |A|$ if and only if G_A is a forest (set of bridges). Also, $A \subseteq B$ implies $r(A) \le r(B)$ and $r(A) = r(E)$ if and only if $c(G_A) = c(G)$.

Proposition 8. *The Tutte polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)}. \tag{5}$$

Proof. Set

$$R(G; u, v) = \sum_{A \subseteq E} u^{r(E)-r(A)} v^{|A|-r(A)},$$

(the Whitney rank-nullity generating function for G). We wish to prove that $T(G; x, y) = R(G; x-1, y-1)$ and shall do this by verifying that $R(G; u, v)$ satisfies the TG-invariant recurrence formula: (i) $R(G; u, v) = 1$ if $E = \emptyset$, (ii) $R(G; u, v) = (u+1)R(G\backslash e; u, v)$ when e is a bridge, (iii) $R(G; u, v) = (v+1)R(G\backslash e; u, v)$ when e is a loop, and (iv) $R(G; u, v) = R(G/e; u, v) + R(G\backslash e; u, v)$ when e is ordinary.

When $E = \emptyset$ we have $R(G; u, v) = 1$.

If $e \notin A$ then

$$r_G(A) = r_{G\backslash e}(A). \tag{6}$$

If $e \in A$ then

$$r_{G\backslash e}(A\backslash e) = \begin{cases} r_G(A) - 1 & \text{if } e \text{ is a bridge,} \\ r_G(A) & \text{if } e \text{ is a loop,} \end{cases} \tag{7}$$

and

$$r_{G/e}(A\backslash e) = r_G(A) - 1 \quad \text{ if } e \text{ is ordinary or a bridge.} \tag{8}$$

Suppose e is a bridge. Then by (6) and (7),

$$\begin{aligned}
R(G;u,v) &= \sum_{A\subseteq E\backslash e} u^{r_G(E)-r_G(A)} v^{|A|-r_G(A)} + \sum_{e\in A\subseteq E} u^{r_G(E)-r(A)} v^{|A|-r_G(A)} \\
&= u \sum_{A\subseteq E\backslash e} u^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)} v^{|A|-r_{G\backslash e}(A)} \\
&\qquad + \sum_{B=A\backslash e} u^{r_{G\backslash e}(E\backslash e)+1-(r_{G\backslash e}(B)+1)} v^{|B|+1-(r_{G\backslash e}(B)+1)} \\
&= (u+1)R(G\backslash e;u,v).
\end{aligned}$$

The case when e is a loop is similarly argued.

When e is ordinary, by (6) and (8),

$$\begin{aligned}
R(G;u,v) &= \sum_{A\subseteq E\backslash e} u^{r_G(E)-r_G(A)} v^{|A|-r_G(A)} + \sum_{e\in A\subseteq E} u^{r_G(E)-r_G(A)} v^{|A|-r_G(A)} \\
&= \sum_{A\subseteq E\backslash e} u^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)} v^{|A|-r_{G\backslash e}(A)} \\
&\qquad + \sum_{B=A\backslash e} u^{r_{G/e}(E\backslash e)+1-(r_{G/e}(B)+1)} v^{|B|+1-(r_{G/e}(B)+1)} \\
&= R(G\backslash e;u,v) + R(G/e;u,v).
\end{aligned}$$

□

It is common to *define* the Tutte polynomial by its subgraph expansion (5), having over the deletion–contraction formulation (1) the advantage of being unambiguously well-defined. On the other hand, it is not apparent from (5) that the coefficients of the Tutte polynomial are non-negative integers, and often it is easier to derive a combinatorial interpretation for an evaluation of the Tutte polynomial by using the deletion–contraction recurrence. Nonetheless, it is easy to read off some evaluations of the Tutte polynomial from its subgraph expansion.

Question 7 Let $G = (V, E)$ be a connected graph. Using the subgraph expansion for $T(G;x,y)$ show the following:

(i) $T(G;1,1) = \#\text{spanning trees}$, $T(G;2,1) = \#\text{spanning forests}$, $T(G;1,2) = \#\text{connected spanning subgraphs}$, and $T(G;2,2) = 2^{|E|} = \#\text{spanning subgraphs}$.

(ii) If $(x-1)(y-1) = 1$ then $T(G;x,y) = (x-1)^{r(E)}y^{|E|}$.

(iii) The generating function for spanning forests of G by number of connected components is given by

$$xT(G;x+1,1) = \sum_{\substack{A\subseteq E\\ n(A)=0}} x^{c(G_A)}.$$

(iv) The generating function for connected spanning subgraphs of G by size is given by

$$y^{|V|-1}T(G;1,y+1) = \sum_{\substack{A\subseteq E\\ c(G_A)=c(G)}} y^{|A|}.$$

Along the hyperbola $(x-1)(y-1) = z$ we have, for graph $G = (V, E)$,

$$\begin{aligned} T(G;x,y) &= (y-1)^{-|V|} \sum_{A \subseteq E} \left(\frac{z}{y-1} \right)^{c(G_A)-c(G)} (y-1)^{|A|+c(G_A)} \\ &= (y-1)^{-r(G)} z^{-c(G)} \sum_{A \subseteq E} z^{c(G_A)} (y-1)^{|A|}. \end{aligned}$$

When $y = 0$ this is the subgraph expansion for the chromatic polynomial that we obtained earlier by inclusion–exclusion. The polynomial $\sum_{A \subseteq E} z^{c(G_A)} w^{|A|}$ is the partition function for the Fortuin–Kasteleyn random cluster model in statistical physics (the normalizing constant for a probability space on subgraphs of G, the probability of $G_A = (V, A)$ depending on both $|A|$ and $c(A)$). This model generalizes the k-state Potts model, which is the case $z = k \in \mathbb{Z}_+$, and whose partition function we have already met in the form of the monochrome polynomial $B(G; k, y)$.

3.4 Coefficients and the spanning tree expansion

A graph invariant is called a *Tutte invariant* if it can be found as some function of the coefficients of $T(G; x, y)$. Thus the property of having at least one edge is a Tutte invariant since $t_{0,0}(G) = 0$ if and only if G has an edge. In fact $|E|$ is itself a Tutte invariant since $T(G; 2, 2) = 2^{|E|}$. Also $r(G) = \max\{i : t_{i,j}(G) \neq 0\}$ and $n(G) = \max\{j : t_{i,j}(G) \neq 0\}$ are Tutte invariants. For another example, from Proposition 4 (ii), a graph G is 2-connected if and only if $t_{1,0}(G) \neq 0$.

Examples of graph invariants that are not Tutte invariants include the degree sequence of G and whether G is planar. A tree on n vertices has Tutte polynomial x^{n-1}, and for $n \geq 3$ there are two trees on n vertices with different degree sequences. Less trivially, there are non-2-isomorphic graphs G and G' which have different degree sequences. Likewise, there is a planar graph G and non-planar graph G' with $T(G; x, y) = T(G'; x, y)$. (See [40, Appendix] for examples.)

In this section we shall give Tutte's 1954 inductive proof that, for a connected graph G, the coefficients $t_{i,j}(G)$ count a certain subset of the spanning trees of G. The interpretation of $t_{i,j}(G)$ when G is not connected follows as an easy consequence of multiplicativity of $T(G; x, y)$ over disjoint unions. A subgraph $G_A = (V, A)$ has $r(A) = r(E)$ and $n(A) = 0$ if and only if G_A is a *maximal spanning forest*, in the sense that no edge can be added to G_A without creating a cycle, i.e., G_A consists of a spanning tree of each connected component of G.

Let $G = (V, E)$ be a connected graph and T a spanning tree of G. Then

(i) for each $e \in E \setminus T$ there is a unique circuit in G contained in $T \cup \{e\}$, which we shall denote by $C_{T,e}$, and

(ii) for each $e \in T$ there is a unique bond contained in $E \setminus T \cup \{e\}$, which we shall denote by $B_{T,e}$.

Put a linear order $<$ on E. Say $E = \{e_1, e_2, \ldots, e_m\}$, where $e_1 < e_2 < \cdots < e_m$.

Definition 9. *Given a spanning tree T of a connected graph G with an ordering of its edges, an edge $e \in T$ is* internally active *with respect to T if e is the least edge in $B_{T,e}$. An edge $e \in E \setminus T$ is* externally active *with respect to T if e is the least edge in $C_{T,e}$. A spanning tree T has* internal activity *i and* external activity *j when there are precisely i internally active edges with respect to T and j externally active edges with respect to T.*

Tutte was led to his spanning tree expansion of the Tutte polynomial of a connected graph by observing that in the recursive definition of $T(G; x, y)$, if one applies deletion and contraction to edges of E in reverse order $e_m, e_{m-1}, \ldots, e_2, e_1$, the result will be an expression for $T(G; x, y)$ as a sum in which each summand is obtained by contracting the elements in some spanning tree T of G and deleting the elements of $E \setminus T$. Moreover, in the process of obtaining this summand the edges contracted as bridges will be precisely the internally active edges with respect to T, and the elements of E deleted as loops will be precisely the externally active edges with respect to T.

Theorem 10. [Tutte, 1954] *Let G be a connected graph with an order on its edges and for each $0 \leq i \leq |V| - 1, 0 \leq j \leq |E| - |V| + 1$ let $t_{i,j}(G)$ denote the number of spanning trees of G of internal activity i and* *external activity j. Then the Tutte polynomial of G is equal to*

$$T(G;x,y) = \sum t_{i,j}(G)x^i y^j.$$

In particular, $t_{i,j}(G)$ is a graph invariant, independent of the ordering of the edges of G.

Proof. We proceed by induction on the number of edges of G.

When there are no edges in G, i.e., $G \cong K_1$, we have $t_{0,0}(G) = 1$ and $t_{i,j}(G) = 0$ for $i + j > 0$.

Let $G = (V, E)$, $E = \{e_1 < e_2 < \ldots < e_m\}$, $m \geq 1$, and assume the assertion holds for connected graphs with at most $m - 1$ edges.

The graphs G/e_m and $G\backslash e_m$ are both connected when e_m is ordinary or a loop, while only G/e_m is connected when e_m is a bridge, but this is fine because we only contract bridges in the recurrence for $T(G;x,y)$. We take $E(G/e_m) = E(G\backslash e_m) = \{e_1 < e_2 < \cdots < e_{m-1}\}$.

(i) Suppose e_m is a bridge. Then e_m is in every spanning tree of G, and a subgraph T is a spanning tree if and only if $e_m \in T$ and T/e_m is a spanning tree of G/e_m. Also, e_m is internally active in every spanning tree T of G, since $B_{T,e_m} = \{e_m\}$, so $t_{0,j}(G) = 0$ for each j. Clearly, for $1 \leq k \leq m - 1$ the edge e_k is internally (externally) active in G with respect to T if and only if it is internally (externally) active in G/e_m with respect to T/e_m. Hence $t_{i,j}(G) = t_{i-1,j}(G/e_m)$ for $i \geq 1$. Applying the inductive hypothesis, we obtain

$$\begin{aligned} T(G;x,y) &= \sum t_{i-1,j}(G/e_m)x^i y^j \\ &= x \sum t_{i-1,j}(G/e_m)x^{i-1} y^j \\ &= xT(G/e_m;x,y) = T(G;x,y). \end{aligned}$$

(ii) Suppose e_m is a loop. Then e_m is in no spanning tree of G, and a subgraph T of G is a spanning tree of G if and only if it is a spanning tree of $G\backslash e_m$. Also e_m is externally active with respect to every spanning tree T of G since $C_{T,e_m} = \{e_m\}$. For $1 \leq k \leq m - 1$ the edge e_k is internally (externally) active in G with respect to T if and only if it is internally (externally) active in $G\backslash e_m$ with respect to the same spanning tree T. Hence $t_{i,j}(G) = t_{i,j-1}(G\backslash e_m)$, so

$$\begin{aligned} \sum_{i,j} t_{i,j}(G)x^i y^j &= y \sum_{i,j} t_{i,j-1}(G\backslash e_m)x^i y^{j-1} \\ &= yT(G\backslash e_m;x,y) = T(G;x,y). \end{aligned}$$

(iii) Suppose e_m is ordinary.

A subset T is a spanning tree of $G\backslash e_m$ if and only if it is a spanning tree of G not containing e_m. If T is a spanning tree of $G\backslash e_m$ with internal activity i and external activity j then it has the same activities as a spanning tree of G, since every other edge precedes e_m and C_{T,e_m} contains an edge other than e_m.

Similarly, T is a spanning tree of G/e_m if and only $T \cup \{e_m\}$ is a spanning tree of G (no cycles in $T \cup \{e_m\}$ can be created by e_m that would not already be in T in the contraction G/e_m). If T is a spanning tree of G/e_m with internal activity i and external activity j then it has the same activities as a spanning tree of G, since every other edge precedes e_m and B_{T,e_m} contains an edge other than e_m since e_m is not a bridge.

It follows that $t_{i,j}(G) = t_{i,j}(G/e_m) + t_{i,j}(G\backslash e_m)$ when e_m is ordinary, and this makes the induction step go through for ordinary edges too. ☐

A more constructive proof that $t_{i,j}(G)$ is equal to the number of spanning trees of G of internal activity i and external activity j was given by Crapo in 1969. See for example [10, ch. 13], and also [11, X.5].

The definition of internal and external activity extends in the obvious way from spanning trees of connected graphs to maximal spanning forests of graphs more generally.

Corollary 11. *Let G be a graph with Tutte polynomial $T(G;x,y) = \sum t_{i,j}(G)x^i y^j$. Then $t_{i,j}(G)$ is equal to the number of maximal spanning forests of G of internal activity i and external activity j.*

Proposition 12. *If $|E(G)| > 0$ then $t_{0,0}(G) = 0$. If $|E(G)| > 1$ then $t_{1,0}(G) = t_{0,1}(G)$.*

Proof. If $E = \{e_1, \ldots, e_m\}$ is non-empty with order $e_1 < \cdots < e_m$, then e_1 is active with respect to any maximal spanning forest F, internally if $e_1 \in F$, externally if $e_1 \notin F$. In particular, $t_{0,0}(G) = 0$.

Note that $t_{1,0}(K_2) = 1, t_{0,1}(K_2) = 0$. Asssume $m \geq 2$. If G has a least two blocks containing at least one edge then we can choose an order on E such that e_1 and e_2 belong to different blocks of G. Then e_1 and e_2 are both active with respect to every maximal spanning forest, and so $t_{1,0}(G) = 0 = t_{0,1}(G)$ in this case.

Suppose then that G is 2-connected. (If there are isolated vertices we can ignore them as the Tutte polynomial is unaffected by their presence or absence.) Let T be a spanning tree of internal activity 1 and external activity 0.

The edge e_1 is active with respect to every spanning tree, and so $e_1 \in T$. This implies $e_2 \notin T$, for otherwise e_2 would also be internally active for T (B_{T,e_2} cannot contain e_1, which belongs to T). So $e_1 \in C_{T,e_2}$, otherwise e_2 would be externally active.

The subgraph $T' = T - \{e_1\} \cup \{e_2\}$ is also a spanning tree of G, and has internal activity 0 and external activity 1 (the edge e_1).

Reversing the argument shows that the map $T \mapsto T'$ is a bijection between trees contributing to $t_{1,0}(G)$ and trees contributing to $t_{0,1}(G)$: if T' is a spanning tree contributing to $t_{0,1}(G)$ then $e_1 \notin T'$ but $e_2 \in T$, and interchanging e_1 and e_2 yields a spanning tree T contributing to $t_{1,0}(G)$. ☐

> **Question 8** Prove Proposition 12 beginning with the fact that $t_{1,0}(G) = 0$ if G is not 2-connected and then inductively by deletion-contraction.

The identities of Proposition 12 are the first of a series of identities proved by Brylawski [18]. If $|E(G)| > k$ then

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{i,j}(G) = 0.$$

Thus if $|E(G)| > 2$ then $t_{2,0}(G) - t_{1,1}(G) + t_{0,2}(G) = t_{1,0}(G)$.

The fact that $T(G; x, y)$ has degree $r(G)$ as a polynomial in x and degree $n(G)$ as a polynomial in y is immediate from the fact that $t_{i,j}(G)$ is the number of maximal spanning forests of internal activity i and external activity j. Choose the edge order $e_1 < e_2 < \cdots < e_m$ so that $e_1, \ldots, e_{r(G)}$ are the edges of a maximal spanning forest: all are internally active, and no edges are externally active when G has no loops. Or, when choosing the edge order so that $e_1, \ldots, e_{n(G)}$ are the edges in the complement of a maximal spanning forest of G, the latter having internal activity 0 provided there are no bridges, and external activity $n(G)$.

3.5 A spanning tree partition of subgraphs

In this section we sketch the relationship between the expansion of the Tutte polynomial by internal–external activities and the subgraph expansion by rank-nullity. In order to do so we rely on many facts given without proof (for which see e.g. [10, ch. 13]).

Let $G = (V, E)$ be a connected graph with a given order on its edges. For a spanning tree T of G, let T^ϵ denote its set of externally active edges and T^ι its set of internally active edges.

The Boolean lattice of subgraphs $2^E = \{A : A \subseteq E\}$ is partitioned into Boolean intervals $[T \setminus T^\iota, T \cup T^\epsilon] = \{A : T \setminus T^\iota \subseteq A \subseteq T \cup T^\epsilon\}$ indexed by spanning trees. Given $A \subseteq E$, we have $n(A) = 0$ (i.e., $r(A) = |A|$) if and only if (V, A) is a forest, and $r(A) = r(E)$ if and only if (V, A) is a connected spanning subgraph. An edge e is *independent* of A if $r(A \cup e) = r(A) + 1$, otherwise e is *dependent*, and $n(A \cup e) = n(A) + 1$. Use the order on E to successively add to A the least edges $e_1, e_2, \ldots, e_{r(E)-r(A)}$ that are independent of A. This creates a connected spanning subgraph $A \cup \{e_1, \ldots, e_{r(E)-r(A)}\}$ containing A.

Similarly, given $A \subseteq E$, by removing edges dependent on A we decrease its nullity, and if $e_1, \ldots, e_{n(A)}$ are chosen to be the least such dependent edges then we obtain a unique subgraph $A \setminus \{e_1, \ldots, e_{n(A)}\}$ of nullity zero, i.e., a spanning forest of G.

If we first add least independent edges to A to make a connected spanning subgraph, and then remove least dependent edges of A we obtain a spanning tree T of G. Likewise, if we first remove the least dependent edges to make a spanning forest and then add the least independent edges we obtain (the same) spanning tree T.

$$\begin{tikzcd}
\begin{matrix}\text{Spanning subgraphs}\\ A \subseteq E\end{matrix} \arrow[r, "\begin{matrix}\text{add least}\\ \text{independent edges}\\ \text{(increase rank)}\end{matrix}"'] \arrow[d, "\begin{matrix}\text{remove least}\\ \text{dependent edges}\\ \text{(decrease nullity)}\end{matrix}"] & \begin{matrix}\text{Connected}\\ \text{spanning subgraphs}\\ r(A) = r(E)\end{matrix} \arrow[d] \\
\begin{matrix}\text{Spanning forests}\\ n(A) = 0\end{matrix} \arrow[r] & \begin{matrix}\text{Spanning trees}\\ |A| = r(A) = r(E)\end{matrix}
\end{tikzcd}$$

This procedure locates which interval $[T \setminus T^{\iota}, T \cup T^{\epsilon}]$ the subset A belongs to. Call A an *internal subgraph* if T has no externally active edges, so A belongs to the interval $[T \setminus T^{\iota}, T]$. (Note that A is internal in this sense if and only if it contains no broken circuit: the least edge removed from a circuit to make a broken circuit contained in A would contribute to the external activity of the tree T containing A.) Similarly, call A an *external subgraph* if it is contained in the interval $[T, T \cup T^{\epsilon}]$ for a spanning tree T with no internally active edges. (If A is external, then $E \setminus A$ contains no broken bonds.)

From the expansion $T(G; x, y) = \sum_{i,j} t_{i,j}(G)x^i y^j$ we see that $T(G; 2, 0)$ is the number of internal subgraphs (this also follows from Whitney's Broken Circuit Theorem) and $T(G; 0, 2)$ is the number of external subgraphs. Moreover, $T(G; 1, 0)$ counts the number of internal trees, and $T(G; 0, 1)$ the number of external trees.

	General	Connected	External
General	$T(G; 2, 2) = 2^{\lvert E \rvert}$	$T(G; 1, 2)$	$T(G; 0, 2)$
Forest	$T(G; 2, 1)$	$T(G; 1, 1)$	$T(G; 0, 1)$
Internal	$T(G; 2, 0)$	$T(G; 1, 0)$	$T(G; 0, 0) = 0$

(We have already seen that $T(G; 2, 0)$ counts acyclic orientations, and for a connected graph $T(G; 1, 0)$ counts acyclic orientations with unique prescribed source. See e.g. [9, Fig. 20] for an interpretation of $T(G; x, y)$ for other values of $x, y \in \{0, 1, 2\}$ in terms of orientations of G. In fact, Las Vergnas [35] gives an interpretation for $2^{i+j} t_{i,j}(G)$ in terms of orientations of G and an order on E.)

Given the spanning tree partition $2^E = \bigcup_T [T \setminus T^{\iota}, T \cup T^{\epsilon}]$ of all subgraphs of G, the subgraph expansion of the Tutte polynomial may be rewritten as follows:

$$\begin{aligned}
T(G; x, y) &= \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{n(A)} \\
&= \sum_T \sum_{A \in [T \setminus T^{\iota}, T \cup T^{\epsilon}]} (x-1)^{|A \cap T^{\iota}|}(y-1)^{|A \cap T^{\epsilon}|} \\
&= \sum_T \sum_{k,\ell} \binom{|T^{\iota}|}{k}(x-1)^k \binom{|T^{\epsilon}|}{\ell}(y-1)^{\ell} \\
&= \sum_T x^{|T^{\iota}|} y^{|T^{\epsilon}|},
\end{aligned}$$

which gives Tutte's spanning tree expansion by internal and external activities.

3.6 Planar graphs

Let $G = (V, E, F)$ be a connected plane graph, with set of faces F, and let $G^* = (V^*, E^*, F^*)$ be its geometric dual. To construct G^*, put a vertex in the interior of each face of G, and connect two such vertices of G^* by edges that correspond to common boundary edges between the corresponding faces of G. If there are several common boundary edges the result is a multiple edge of G^*.

Recall that for a spanning tree T of G, T^ϵ denotes its set of externally active edges and T^ι its set of internally active edges.

We identify V^* with F, E^* with E, and F^* with V.

Proposition 13. *There is a bijection $T \mapsto T^*$ between spanning trees of G and spanning trees of G^* which switches internal and external activities. Specifically, $T^* = E \setminus T$, and $t_{i,j}(G^*) = t_{j,i}(G)$.*

Proof. The set of edges T^* in the dual G^* corresponding to the set of edges $E \setminus T$ in G together connect all the faces of G, since T has no cycles. (A cycle of edges would be required to separate one set of faces from another, their edges forming a simple closed curve partitioning the plane into inside and outside. If there are no cycles the plane remains in one piece.) Also, T^* does not contain a cycle, for otherwise it would separate some vertices in G inside the cycle from vertices outside, and this is impossible because T is spanning and its edges are disjoint from T^*.

This shows that T^* is a spanning tree of G^*.

Recall that $C_{T,e}$ denotes the unique circuit contained in $T \cup \{e\}$ and $B_{T,e}$ the unique cocircuit (bond) contained in $T \setminus \{e\}$. Given an edge $e \in T$ we have $B_{T,e} = C_{T^*,e}$ (why?). Dually, given an edge $e \in E \setminus T$ we have $C_{T,e} = B_{T^*,e}$. Consequently $T^\iota = (T^*)^\epsilon$ and $T^\epsilon = (T^*)^\iota$, from which it follows that $t_{i,j}(G^*) = t_{j,i}(G)$. □

Corollary 14. *If G is a connected planar graph with dual G^* then $T(G^*; x, y) = T(G; y, x)$.*

Note that a bridge in G is a loop in G^*, a loop in G is a bridge in G^*, and that deleting (contracting) an edge in G corresponds to contracting (deleting) an edge in G^*. In other words, $(G/e)^* \cong G^* \backslash e$ and $(G \backslash e)^* \cong G^*/e$. From these properties, that $T(G^*; x, y) = T(G; y, x)$ also follows from the deletion-contraction recurrence for the Tutte polynomial.

More generally, a subgraph of G on edges $A \subseteq E$ has no circuits (i.e., is a forest) if and only if the subgraph in the dual G^* on edges $E \setminus A$ is connected. If there is a cycle in A then its edges form the boundary of a simple closed curve in the plane, inside which lies at least one vertex of G^* (corresponding to a face enclosed by the cycle) and outside of which lies another vertex of G^*. Likewise, the edges of A form a connected subgraph of G if and only if the edges of $E \setminus A$ form a forest of G^*: any cycle in G^* has to cross an edge of a connected subgraph A.

In the terminology of the Section 3.5, an edge $e \in E \setminus A$ is independent of A in G if and only if it is a dependent edge of $E \setminus A$ in G^*. (And the dual statement holds: an edge $e \in A$ is a dependent edge of G if and only if it is an independent edge of $E \setminus A$.) The maximum number k of edges $e_1, \dots, e_k$ such that e_i is independent of $A \cup \{e_1, \dots, e_{i-1}\}$ for each $1 \le i \le k$ is equal to $r_G(E) - r_G(A)$, which is therefore equal to the maximum number k of edges $e_1, \dots, e_k$ so that e_i is dependent on $E \setminus (A \cup \{e_1, \dots, e_i\})$ for each $1 \le i \le k$.

Question 9

(i) Prove that the rank and nullity functions of a planar graph and its dual are related by $r_{G^*}(A) = n_G(E) - n_G(E \setminus A) = |A| - r_G(E) + r_G(E \setminus A)$, and $n_{G^*}(A) = r_G(E) - r_G(E \setminus A) = |A| - n_G(E) + n_G(E \setminus A)$.

(ii) Deduce that $T(G^*; x, y) = T(G; y, x)$ by using the subgraph expansion of the Tutte polyomial.

The Tutte polynomial of a planar graph can be expressed in terms of topological properties of its *medial graph*, as we shall shortly describe. This is the starting point for the fruitful cross-fertilization of graph theory and knot theory - for a 4-regular plane graph can be read as the shadow of a knot in 3-dimensional space.

First we refresh the memory conerning medial graphs. To form the medial graph $m(G)$ of a connected plane graph G that has at least one edge first place a vertex v_e into the interior of each edge e of G. Then, for each face F of G, join v_e and v_f by an edge lying in F if and only if the edges e and f are consecutive on the boundary of F. The medial graph $m(G)$ is 4-regular, as each face creates two adjacencies for each edge on its boundary. The faces of $m(G)$ divide naturally into two types: those that contain vertices of G (vertex-faces), and those corresponding to faces of G (face-faces). Vertex-faces will be coloured black and face-faces coloured white. See Figure 2.

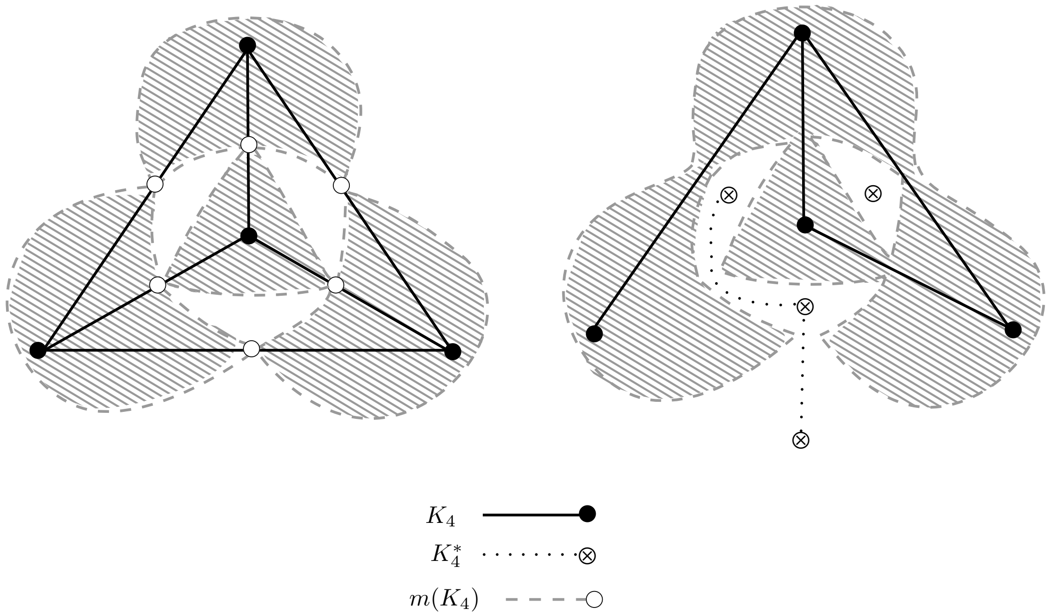

Figure 2: On the left, K_4 and its medial graph, with faces containing vertices of G shaded black. A white, black or crossing transition at each vertex of $m(K_4)$ together determine a set of closed Eulerian walks partitioning the edge set. The example on the right illustrates a white-black Eulerian partition of $m(K_4)$ into two cycles, with associated subgraphs of K_4 (edges where there are black transitions) and its plane dual K_4^* (edges where there are white transitions).

If G^* is the planar dual of G then $m(G^*) \cong m(G)$ (if $e \mapsto e^*$ is the duality mapping between edges of G and edges of G^* then e and f are consecutive edges of a face in G if and only if e^* and f^* are consecutive edges in a face of G^*).

The plane graph G is the *black face graph* of $m(G)$, i.e., the graph whose vertices are the black faces of $m(G)$ and whose edges join two black faces of $m(G)$ that share a vertex. The plane graph G^* is the white face graph of $m(G)$.

Forming the black face graph is inverse to the medial construction. A 4-regular connected plane graph H has bipartite dual graph so we can always 2-colour the faces of H properly with colours black and white, making the exterior face white. If $G(H)$ is the black face graph of H then $m(G(H)) = G$.

A walk in a graph is an alternating sequence of vertices and edges that starts and finishes with the a vertex, with the property that consecutive vertices are the endopints of an edge joining them. A walk is closed if its first and last elements are equal, and *Eulerian* if it uses each edge at most once.

We are going to look in more detail at the Eulerian walks determined by a transition system. Cyclically rotating the sequence of vertices and edge in a closed Eulerian walk yields another closed Eulerian walk. Up to equivalence this determines an *oriented Eulerian cycle*. If we count two Eulerian walks to be equivalent if they are the same up to both rotation and reversal of the sequence of vertices of edges then we have determined an *Eulerian cycle*. (Note that an Eulerian cycle need not be a circuit as it may pass through a vertex more than once.)

The case of loops is a little awkward, and to accommodate them properly when considering Eulerian cycles it is safest to resort to the use of half-edges and to define walks as appropriate types of sequences of half-edges. For us it suffices to remark that a loop can be traversed in two ways, which gives rise to two non-equivalent Eulerian cycles that use it, except when the cycle is just the loop by itself, when it counts just as one cycle.

An *Eulerian partition* of an Eulerian graph is a collection of Eulerian cycles such that each edge belongs to exactly one cycle. An *Eulerian tour* is an Eulerian cycle spanning all the vertices (an Eulerian partition with just one cycle).

A matching of the four edges incident with a vertex of $m(G)$ into two pairs $\{e_1, f_1\}$ and $\{e_2, f_2\}$ is called a *transition*, and when all vertices have transitions we speak of a *transition system*. A transition system determines a set of closed Eulerian walks, where two consecutive edges of a walk form one of the pairs in a transition. A transition is *white* if the edges e_1 and f_1 are consecutive edges of a white face of $m(G)$ (and hence so are e_2 and f_2). Likewise, a transition is *black* if the edges e_1 and f_1 are consecutive edges of a black face of $m(G)$ (and so are e_2 and f_2). Otherwise, a transition neither white nor black is called a *crossing* transition. See Figure 3, where a transition $\{e_1, f_1\}$, $\{e_2, f_2\}$ at a vertex is represented by suppressing the vertex and, for $i = 1, 2$, joining e_i and f_i to make a single edge whose endpoints are the two other vertices incident with these original edges. (If this is done for every vertex then eventually one obtains a set of curves, one for each of the closed Eulerian walks determined by the transition system, and these curves are non-intersecting if there are no crossing transitions.)

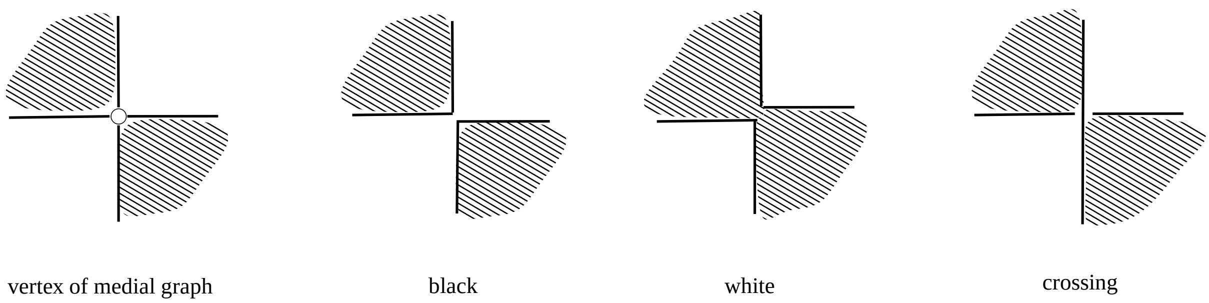

Figure 3: Three types of transition at the vertex of a 4-regular graph. An Eulerian walk matches the four edges incident with a vertex into two pairs. (The vertex is deleted, and each matched pair of edges is joined to make a connected line.)

A *white-black* Eulerian partition is one in which very transition is white or black, and a white-black Eulerian tour is defined likewise. Godsil and Royle call a white-black Eulerian partition a *bent* Eulerian partition, as following the tour you turn a bend at every corner, never going straight on. Kauffman [31] and [30] calls a white-black Eulerian tour a *Jordan-Euler trail*, because the trail (i.e., tour, in our terminology) forms a Jordan curve in the plane when its transitions are represented as in Figure 3. Since the medial graph comes with a proper face 2-colouring with exterior face white, the inside of the Jordan curve formed by an Eulerian tour is black and its outside is white.

Lemma 15. *Let $m(G)$ be the medial graph of plane graph G. Then there is a bijection between white-black Eulerian tours of $m(G)$ and spanning trees of G.*

Proof. The graph G is the black face graph of $m(G)$, and its dual G^* the white face graph of $m(G)$. Given a white-black Eulerian tour of $m(G)$, define a graph T^* on white faces by joining two white faces (vertices of G^*) by an edge if they meet at a vertex at which there is a black transition. Likewise, define a graph T on black faces by joining two black faces (vertices of G) by an edge if they meet at a vertex at which there is a white transition. Since the edges of the Euler tour form a single Jordan curve it follows that T corresponds to a spanning tree of G and T^* to a spanning tree of G^*.

Conversely, give a spanning tree T of G (viewed as a spanning tree of the black face graph of $m(G)$) the inverse operation of assigning white transitions to vertices of $m(G)$ common to a pair of black faces adjacent in T and black transitions to the other vertices of $m(G)$ yields an Eulerian tour of $m(G)$. (Black transitions are at vertices common to adjacent white faces in T^* viewed as a spanning tree of the white face graph of $m(G)$. The spanning tree T^* of G^* has edge set under the duality correspondence $e \mapsto e^*$ equal to $E(G) \setminus T$.) ☐

In view of Lemma 15, we may identify an Eulerian tour T of $m(G)$ with a spanning tree T of G. Following Kauffman [31], we define activities for white-black Eulerian tours, using Tutte's spanning tree activities as a guide. Suppose G has m edges. Each vertex of $m(G)$ corresponds to a unique edge of G (and of G^*). Label the vertices of $m(G)$ by $[m]$. Given an Eulerian tour of $m(G)$, suppose we replace the transition at vertex i by its opposite, i.e., swapping white for black or vice versa.

This produces two component Eulerian cycles. Say two vertices *interact* if they belong to different Eulerian cycles. The vertex labelled i is *active* if it has the least label amongst all interacting vertices.

We then have
$$T(G;x,y) = \sum_{\text{white-black Eulerian tours } T \text{ of } m(G)} x^{wa(T)}y^{ba(T)},$$
where $wa(T)$ and $ba(T)$ respectively denote the number of active vertices at which T has a white or black transition.

We can develop this correspondence from Eulerian tours and spanning trees to a one-to-one correspondence between white-black Eulerian partitions of $m(G)$ and spanning subgraphs of G. Identifying edges of G with vertices of $m(G)$, to $A \subseteq E(G)$ there corresponds a white-black Eulerian partition of $m(G)$, which we shall denote by $\mathcal{E}_A$, whose white transitions are at vertices in A and whose black transitions are at vertices in $E \setminus A$.

Each Eulerian cycle C of $\mathcal{E}_A$ is a Jordan curve, whose interior may contain other Eulerian cycles of the partition. Suppose any Eulerian cycles that lie inside C are contracted one by one to points in the plane, contracting first those that contain no other cycle, and then these points are deleted: this leaves the interior of C a single colour, either white or black, and we call C an *inwardly white* or *inwardly black* Eulerian cycle of $\mathcal{E}_A$ accordingly. (This is not to cast aspersions on its moral rectitude.) A connected component of (V, A) corresponds to an inwardly black Eulerian cycle in $\mathcal{E}_A$, while a connected component of the subgraph of G^* spanned by $E(G) \setminus A$ not containing the vertex corresponding to the exterior white face corresponds to an inwardly white Eulerian cycle of $\mathcal{E}_A$.

Theorem 16. *Let $G = (V, E)$ be a plane graph and $m(G) = (E, L)$ its medial graph. For each $A \subseteq E$ let $\mathcal{E}_A$ denote the white-black Eulerian partition of $m(G)$ whose white transitions occur at vertices in A and black transitions at vertices in $E \setminus A$. Then*
$$T(G; x+1, y+1) = \sum_{A\subseteq E} x^{b(\mathcal{E}_A)-1}y^{w(\mathcal{E}_A)},$$
where $b(\mathcal{E}_A)$ is the number of inwardly black Eulerian cycles in the Eulerian partition $\mathcal{E}_A$ of $m(G)$ and $w(\mathcal{E}_A)$ the number of inwardly white Eulerian cycles.

Proof. By the subgraph expansion for the Tutte polynomial,
$$T(G; x+1, y+1) = \sum_{A\subseteq E} x^{r_G(E)-r_G(A)}y^{n_G(A)}.$$
Consider a connected plane graph $G = (V, E, F)$ with dual G^* identified with (F, E, V). For $A \subseteq E$, the subgraph (V, A) of G has $r_G(E) - r_G(A) + 1$ components, and the subgraph $(F, E \setminus A)$ of G^* has $r_{G^*}(E) - r_{G^*}(E \setminus A) + 1 = n_G(A) + 1$ components. The result now folllows by the correspondence already described between connected components of (V, A) and inwardly-black Eulerian cycles, and between components of $(F, E \setminus A)$ and inwardly-white Eulerian cycles (with the exception of the one component that contains the vertex of G^* corresponding to the outer white face of $m(G)$). $\square$

As a corollary we have the following result due to Martin [37, 38].

Corollary 17. *Let G be a connected plane graph and $m(G)$ its medial graph. Then the number of white-black Eulerian partitions with c components is equal to the coefficient of x^{c-1} in $T(G; x+1, x+1)$.*

Corollary 17 can be proved directly by deletion-contraction and appeal to the Recipe Theorem: deleting an edge of G is to choose a black transition at the corresponding vertex of $m(G)$, while contraction makes a white transition. (Duality of deletion and contraction is here observed in that white transitions correspond to edges of G^*.)

Question 10 Let G be a plane graph and $m(G)$ its medial graph with the usual colouring of its vertex-faces black and face-faces white. Suppose $m(G)$ is given an orientation so that each black face has an anticlockwise rotation (each arc has a black face to its left).

(i) Explain why a white-black Eulerian partition of the undirected graph $m(G)$ can be viewed as a partition of the oriented graph $m(G)$ into *directed* Eulerian cycles.

(ii) Show that an inwardly black Eulerian cycle of $m(G)$ corresponds to a directed Eulerian cycle traversed in an anticlockwise sense in the plane, and an inwardly white Eulerian cycle to a clockwise directed Eulerian cycle.

(iii) Reformulate Theorem 16 for the oriented medial graph in terms of partitions into directed Eulerian cycles.

This section has given but a surface glimpse of the deep relationship of the Tutte polynomial to knot invariants. We refer the interested reader to [53] and the references contained therein.

Before moving on let's just finish with an informal description of a couple of further results as they are particularly relevant to what we have been looking at and do not require the introduction of additional concepts.

Jaeger's *transition polynomial* is defined by

$$Q(m(G);\alpha,\beta,\gamma;z) = \sum_{\text{Eulerian partitions}} \alpha^{\#\,\text{white}}\beta^{\#\,\text{black}}\gamma^{\#\,\text{crossing}}z^{\#\,\text{components}},$$

where the sum ranges over all $3^{|E|}$ Eulerian partitions (equivalently, transition systems), "# white" denotes the number of white transitions etc. and # components the number of Eulerian cycles in the Eulerian partition. Note that $Q(m(G);\alpha,\beta,\gamma;1) = (\alpha+\beta+\gamma)^{|E|}$. Corollary 17 involves the case $\alpha=\beta=1$, stating that$Q(m(G);1,1,0;z) = T(G;1+z,1+z)$. Jaeger [27] proved more generally that if $\gamma=0$ and $\alpha\beta\neq 0$ then, for a connected plane graph $G=(V,E,F)$,

$$Q(m(G);\alpha,\beta,0;z) = \alpha^{|F|-1}\beta^{|V|-1}T(G;1+\frac{\alpha}{\beta}z,1+\frac{\beta}{\alpha}z).$$

What about transition systems where $\gamma\neq 0$? If we ask that every vertex transition is crossing, i.e., $\alpha=\beta=0$, then there is just one Eulerian partition. Whether this consists of just one component turns out to depend on whether G has any *bicycles*, which we shall define and study in Section 3.7.1. See e.g. [23] and [5] for a proof of the following result:

Theorem 18. *Let $G=(V,E)$ be a connected plane graph and $m(G)$ its medial graph. Then*

$$T(G;-1,-1) = (-1)^{|E|}(-2)^{c-1},$$

where c is the number of components in the Eulerian partition of $m(G)$ where each transition a crossing. In particular, there is an Eulerian tour of $m(G)$ whose transitions are all crossings if and only if G has an odd number of white-black Eulerian tours.

A curious counterpart to Theorem 18 is the following result of Las Vergnas [36]:

Theorem 19. *Let G be a connected plane graph and $m(G)$ its medial graph. Then*

$$T(G;3,3) = K2^{c-1},$$

where c is the number of Eulerian cycles in the everywhere-crossing Eulerian partition of $m(G)$ and K is an odd integer.

(Since $T(G;3,3)\equiv T(G;-1,-1) \pmod 4$, we know from Theorem 18 that $T(G;3,3)$ is odd when there is an Eulerian tour all of whose transitions are crossing, and that $T(G;3;3)$ is singly even if there are exactly two component cycles in the everywhere-crossing Eulerian partition. The content of Theorem 19 is to extend these observations to any number of component cycles.)

Corollary 17 with $x = 2$ provides a combinatorial interpretation of sorts for $T(G;3,3)$ (in terms of 2-coloured Eulerian cycle partitions of $m(G)$). Las Vergnas gave one in terms of Eulerian orientations of $m(G)$:

$$2\,T(G;3,3) = \sum_{\substack{\text{Eulerian orientations}\\ \text{of } m(G)}} 2^{\#\text{ saddle vertices}},$$

where a saddle vertex has edges directed "in, out, in, out" in cyclic order. (The one interpretation can be derived from the other, as shown in [32].)

Question 11 Suppose that as for Question 11 the medial graph $m(G)$ of plane graph G is given an orientation so that black faces are traversed anticlockwise.

(i) Show that the Eulerian partition of the undirected medial graph with all transitions crossing corresponds to a partition of the directed medial graph into cycles whose edges alternate in direction when traversed.

(ii) Reformulate Theorem 18 in terms of the oriented medial graph.

Choosing the weights $\alpha = 0, \beta = 1, \gamma = -1$ in the transition polynomial gives the *Penrose polynomial*, about whose fascinating properties and suggestive intersection with the Tutte polynomial at precisely the locus of the Four Colour Theorem see e.g. [5].

It is possible to extend consideration from the medial graph of plane graphs to 4-regular graphs embedded in surfaces of higher genus, with the Tutte polynomial of Theorem 16 being replaced by the coloured Tutte polynomial of Bollobás and Riordan. See [32].

The Penrose polynomial, and transition polynomials more generally, has been extended from plane graphs to dual pairs of binary matroids, via *isotropic systems*. See in particular Bouchet [15], [16], Aigner [1], [3], Aigner and Mielke [2], Las Vergnas [33], [34], Ellis-Monaghan [41], [42], [43], Bollobás [14], Jaeger [27]. The Martin polynomial was discovered independently (in the form of the "circuit partition polynomial") in the context of DNA sequencing [6]. The latter paper uses the encoding of Eulerian partitions of planar 4-regular graphs by interlacement graphs to define an interlace polynomial defined for an arbitrary graph, for which see [7], [8].

3.7 Tension-flows

We now return to general graphs, not necessarily planar, and explore a further aspect of the duality between tensions and flows. We begin with bicycles, for these are related to the all-crossing Eulerian partitions of medial graphs that we encountered in the previous section.

First though let's revisit some facts about tensions and flows to prepare the ground for bicycles.

Let $G = (V, E)$ be a graph, A an Abelian group of order k, and $\mathcal{C}$ the set of A-flows of G and its orthogonal complement $\mathcal{C}^\perp$ the set of A-tensions of G.

The monochrome polynomial $B(G;k,y)$ of G was defined just before Proposition 17 in the chapter on the chromatic polynomialin terms of vertex k-colourings, but we can write it in terms of A-tensions as follows:

$$k^{-c(G)}B(G;k,y) = \sum_{\mathbf{z}\in\mathcal{C}^\perp} y^{|E|-|\mathrm{supp}(\mathbf{z})|}. \tag{9}$$

In coding theory $|\mathrm{supp}(\mathbf{z})|$ is called the *Hamming weight* of the vector $\mathbf{z}$ and the polynomial on the right-hand side of (9) is known as the (Hamming) *weight enumerator* of the code $\mathcal{C}^\perp$.

By deletion-contraction and the Recipe Theorem we have seen that

$$B(G;k,y) = k^{c(G)}(y-1)^{r(G)}T(G;\frac{y-1+k}{y-1},y). \tag{10}$$

A code over a field $\mathbb{F}$ is a special type of matroid, namely one that is representable over $\mathbb{F}$. The point $(\frac{y-1+k}{y-1}, y)$ lies on the hyperbola $(x-1)(y-1) = k$. Greene [25] was first to make the connection between the Tutte polynomial and linear codes over a field of k elements, proving that the Tutte polynomial of the matroid of a code specializes on the hyperbola $(x-1)(y-1)=k$ to the weight enumerator of the code (effectively, identity (10) generalized to codes/representable matroids).

The dual version of the monochrome polynomial (the weight enumerator for A-tensions (9)) is the weight enumerator for A-flows:

$$C(G;k,x)=\sum_{\mathbf{z}\in\mathcal{C}} x^{|E|-|\mathrm{supp}(\mathbf{z})|}=(x-1)^{n(G)}T(G;x,\frac{x-1+k}{x-1}). \tag{11}$$

(This identity can be proved by an inductive deletion-contraction argument, as for the monochrome polynomial.) Thus by identities (10) and (11) we have

$$B(G;k,y)=k^{|V(G)|-|E(G)|}(y-1)^{|E(G)|}C(G;k,\frac{y-1+k}{y-1}), \tag{12}$$

which amounts to MacWilliams identity in coding theory.

3.7.1 Bicycles

In this section we take $A=\mathbb{F}_2$ in the incidence mapping $D: A^E\to A^V$ for a graph $G=(V,E)$. As usual we write $\mathcal{C}=\ker D$ and $\mathcal{C}^\perp=\operatorname{im} D^\top$. In the chapter on tensions and flows we saw that vectors in $\mathcal{C}$ are characteristic vectors of Eulerian subgraphs of G (sometimes just called cycles or even subgraphs of G), and that vectors in $\mathcal{C}^\top$ are characteristic vectors of cutsets of G.

An Eulerian subgraph meets a cutset in an even number of edges (by orthogonality of flows and tensions, and by definition when considering cuts comprising edges from $\{v\}$ to $V\setminus\{v\}$, these vertex-cuts together spanning all cuts).

We identify a subset of edges of G with its characteristic vector.

A vector $\mathbf{x}$ in the intersection $\mathcal{C}\cap\mathcal{C}^\perp$ is called a *bicycle* of G, and is self-orthogonal, i.e., $\mathbf{x}^\top\mathbf{x}=0$. So a bicycle has an even number of edges. A bicycle is an Eulerian subgraph that meets every other Eulerian subgraph in an even number of edges (as well as every cut in an even number of edges). Alternatively, a bicycle is a cut that meets every other cut in an even number of edges (as well as meeting every Eulerian subgraph in an even number of edges).

In short, a bicycle is a cutset that is also an Eulerian subgraph of G. In particular, if G is itself a bipartite Eulerian graph then E (the all-one vector) is a bicycle.

For more about bicycles see Sections 14.15-16 and 15.7 in [23], and for the usefulness of bicycles in relation to knots see Chapter 17 of the same reference.

Theorem 20. *Let e be the edge of a graph G. Then precisely one of the following holds:*

(i) e belongs to a bicycle,

(ii) e belongs to a cut B such that $B\setminus\{e\}$ is Eulerian,

(iii) e belongs to an Eulerian subgraph C such that $C\setminus\{e\}$ is a cut.

Proof. Suppose $e\in E(G)$ and $\mathbf{e}$ is its indicator vector in $\mathbb{F}_2^E$. If e belongs to a bicycle with indicator vector $\mathbf{x}$ then $\mathbf{x}^\top\mathbf{e}\neq 0$ and therefore $e\notin(\mathcal{C}\cap\mathcal{C}^\perp)^\perp=\mathcal{C}^\perp+\mathcal{C}$. If e does not belong to a bicycle then $\mathbf{e}$ is orthogonal to all vectors in $\mathcal{C}\cap\mathcal{C}^\perp$ and so $\mathbf{e}\in\mathcal{C}+\mathcal{C}^\perp$. In other words, e is either contained in a bicycle or e is the symmetric difference of an Eulerian subgraph and a cutset.

In any representation of e as the symmetric difference of an Eulerian subgraph and a cut, either e will always belong to the Eulerian subgraph, or e will always belong to the cut. For suppose that $\mathbf{e}=\mathbf{z}+\mathbf{y}=\mathbf{z}'+\mathbf{y}'$ where $\mathbf{z},\mathbf{z}'\in\mathcal{C}$ and $\mathbf{y},\mathbf{y}'\in\mathcal{C}^\perp$. Then $\mathbf{z}+\mathbf{z}'\in\mathcal{C}$ and $\mathbf{y}+\mathbf{y}'\in\mathcal{C}^\perp$ so $\mathbf{z}+\mathbf{z}'=\mathbf{y}+\mathbf{y}'$ is a bicycle. Since e does not belong to a bicycle, it must belong to both or neither of $\mathbf{z}$ and $\mathbf{z}'$,and to neither or both of $\mathbf{y}$ and $\mathbf{y}'$, respectively (since $\mathbf{e}=\mathbf{z}+\mathbf{y}$). ☐

An edge e of G is of *bicycle-type*, *cut-type* or *flow-type* according as (i), (ii) or (iii) holds in the statement of Theorem 20, respectively. This is known as the principal tripartition of the edges of G.

A bridge is an edge of cut-type [take cut $B = \{e\}$ in (ii)] and a loop is an edge of flow-type [take Eulerian subgraph $\{e\}$ in (iii)].

If G is planar then edges of bicycle-type in G remain of bicycle-type in G^*. By flow–tension duality, edges of cut-type in G are edges of flow-type in G^*, and similarly edges of flow-type in G^* are edges of cut-type in G^*.

Question 12 Let G be a graph with incidence matrix D and $Q = DD^T$ be the Laplacian matrix of G.
For edge $e = uv$, let $\mathbf{y} \in \mathbb{F}_2^V$ be the vector with 1 in the places indexed by u and v and 0 elsewhere (i.e., the column of D indexed by e, which when read in $\mathbb{F}_2$ is a $0-1$ vector).
Prove the following:

(i) If $Q\mathbf{x} = \mathbf{y}$ has no solution then e is of bicycle-type.

(ii) If $Q\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x}$ with $\mathbf{x}^T Q\mathbf{x} \neq 0$, then e is of cut-type.

(iii) If $Q\mathbf{x} = \mathbf{y}$ has a solution $\mathbf{x}$ with $\mathbf{x}^T Q\mathbf{x} = 0$, then e is of cut-type.

Consequently, whether an edge is of bicycle-, cut- or flow-type can be decided in polynomial time.

Lemma 21. *Let G be a graph with bicycle space of dimension d, and e an edge of G. The following table gives the dimension of the bicycle space of G/e and $G\backslash e$.*

Type of e	G/e	$G\backslash e$
Bridge or loop	d	d
Bicycle-type	$d-1$	$d-1$
Cut-type, not bridge	d	$d+1$
Flow-type, not loop	$d+1$	d

Proof. A bridge belongs to no cycle and hence to no Eulerian subgraph, and therefore to no bicycle. So any bicycle of G is a bicycle of $G\backslash e$. Conversely, a bicycle of $G\backslash e$ is also a bicycle of G. Likewise, bicycles of G/e correspond to bicycles of G.

Similarly, a loop belongs to no cut and hence to no bicycle, so bicycles of G are bicycles of $G\backslash e$, and conversely. For a loop we have $G/e \cong G\backslash e$.

For an ordinary edge e we shall find the following two observations useful:

(i) If e is not a loop and belongs an Eulerian subgraph C, then $C \setminus \{e\}$ is neither an Eulerian subgraph of G nor of $G\backslash e$. On the other hand, $C \setminus \{e\}$ is an Eulerian subgraph of G/e.

(ii) Dually, if e is not a bridge and belongs to a cut B, then $B \setminus \{e\}$ is neither a cut of G nor of G/e. On the other hand, $B \setminus \{e\}$ is a cut of $G\backslash e$.

Suppose then that e is an ordinary edge. We distinguish the three cases of the principal tripartition:

(a) e belongs to a bicycle A.

By (i) and (ii), $A \setminus \{e\}$ is not a bicycle of G, $G\backslash e$ or G/e. On the other hand, any bicycle of G which does not contain e remains a bicycle of $G\backslash e$ and G/e. Hence the bicycle spaces of $G\backslash e$ and of G/e both correspond to the subspace of bicycles of G that do not contain e, and their dimensions are therefore 1 less than the bicycle dimension of G.

(b) e belongs to a cut B, such that $B \setminus \{e\}$ is an Eulerian subgraph of G.

By (ii), the set $B \setminus \{e\}$ is a cut of $G\backslash e$, but not of G or G/e. Hence $B \setminus \{e\}$ is a bicycle of $G\backslash e$, but not of G or G/e. The effect is to increase the dimension of the bicycle space of $G\backslash e$ by 1. All bicycles of G are bicycles of $G\backslash e$ since e is of cut-type, and so bicycles of $G\backslash e$ are bicycles of G together with symmetric difference of bicycles of G with the fixed set $B \setminus \{e\}$. On the other hand, the dimension of the bicycle space of G/e coincides with that of G, all bicycles of G being bicycles of G/e, and no others.

(c) e belongs to an Eulerian subgraph C such that $C \setminus \{e\}$ is a cut.

By (i), the set $C \setminus \{e\}$ is an Eulerian subgraph of G/e, but not of G or $G\backslash e$. Hence $C \setminus \{e\}$ is a bicycle of G/e, but not of G or $G\backslash e$. Similarly to case (b), this implies the dimension of the bicycle space of G/e is 1 more than that of G, while $G\backslash e$ has the same bicycle dimension as G.

□

Lemma 22. *Let $G = (V, E)$ be a graph with bicycle space of dimension $b(G)$, and let e be an edge of G. Then the graph invariant*

$$f(G) = (-1)^{|E|}(-2)^{b(G)}$$

satisfies

$$f(G) = \begin{cases} (-1)f(G/e) & e \text{ a bridge,} \\ (-1)f(G\backslash e) & e \text{ a loop,} \\ f(G/e) + f(G\backslash e) & e \text{ ordinary.} \end{cases}$$

Proof. We use Lemma 21.

If e is a bridge or loop then the bicycle spaces of G/e, $G\backslash e$ and G are all of the same dimension, and this implies the first two cases.

Suppose e is ordinary. If e is of cut-type then

$$\begin{aligned} f(G/e) + f(G\backslash e) &= (-1)^{|E|-1}(-2)^{b(G)} + (-1)^{|E|-1}(-2)^{b(G)+1} \\ &= (-1)^{|E|}(-2)^{b(G)}. \end{aligned}$$

If e is of flow-type then

$$\begin{aligned} f(G/e) + f(G\backslash e) &= (-1)^{|E|-1}(-2)^{b(G)+1} + (-1)^{|E|-1}(-2)^{b(G)} \\ &= (-1)^{|E|}(-2)^{b(G)}. \end{aligned}$$

If e belongs to a bicycle then

$$f(G/e) + f(G\backslash e) = 2(-1)^{|E|-1}(-2)^{b(G)-1} = (-1)^{|E|}(-2)^{b(G)}.$$

□

By the Recipe Theorem (Theorem 5) we obtain the following polynomial-time computable evaluation of the Tutte polynomial:

Theorem 23 ([46])**.** *Let $G = (V, E)$ be a graph and let $b(G)$ denote the dimension of its bicycle space. Then $(-1)^{|E|}(-2)^{b(G)} = T(G; -1, -1)$.*

Question 13
Prove that a connected graph G has no non-trivial bicycles if and only if G has an odd number of spanning trees.

3.7.2 $\mathbb{Z}_3$-tension-flows

In this section we take $A = \mathbb{F}_3$ (additive group isomorphic to $\mathbb{Z}_3$) and consider the intersection of the space of $\mathbb{Z}_3$-flows and the space of $\mathbb{Z}_3$-tensions. If $D : \mathbb{F}_3^E \to \mathbb{F}_3^V$ is the incidence mapping, and we let $\mathcal{C} = \ker D$, so that $\mathcal{C}^\perp = \operatorname{im} D^\top$, then we shall call a vector in $\mathcal{C} \cap \mathcal{C}^\perp$ a $\mathbb{Z}_3$-*tension-flow*. In other words, a $\mathbb{Z}_3$-tension-flow is both a $\mathbb{Z}_3$-tension and a $\mathbb{Z}_3$-flow, and is self-orthogonal in $\mathbb{F}_3^E$. (In this terminology we could have called bicycles $\mathbb{Z}_2$-*tension-flows.*)

Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity. In [28] Jaeger proved by a deletion-contraction argument that $T(G;\omega,\omega^2) = \pm\omega^{|E|+\dim\mathcal{C}}(i\sqrt{3})^{\dim(\mathcal{C}\cap\mathcal{C}^\perp)}$, using the *principal quadripartition* of the edges of a graph (a generalization to flows and tensions over finite fields of characteristic $\neq 2$ of the principal tripartition). Gioan and Las Vergnas [22] provide a linear algebra proof that has the benefit of determining the sign. It is this latter proof that we shall present here.

Recall that we say vectors $\mathbf{y}$ and $\mathbf{z}$ are orthogonal if $\mathbf{y}^\top\mathbf{z} = 0$. A *self-orthogonal* vector (also called an *isotropic* vector) is a vector $\mathbf{z}$ with $\mathbf{z}^\top\mathbf{z} = 0$.

Lemma 24. *Let $\mathcal{C}$ be a finite-dimensional vector space over a field of characteristic not equal to 2. Then $\mathcal{C}$ has an orthogonal basis.*

Proof. Let $\{\mathbf{z}_1, \dots, \mathbf{z}_d\}$ be a basis for $\mathcal{C}$. If there is an index $1 \leq i \leq d$ such that $\mathbf{z}_i$ is not self-orthogonal then reindex in such a way that $i = 1$ and set $\mathbf{z}'_1 = \mathbf{z}_1$. Otherwise, if there is an index $2 \leq i \leq d$ such that $\mathbf{z}_1 + \mathbf{z}_i$ is not self-orthogonal then set $\mathbf{z}'_1 = \mathbf{z}_1 + \mathbf{z}_i$. In both cases update $\mathbf{z}_j$ as $\mathbf{z}_j - \frac{\mathbf{z}_1'^\top\mathbf{z}_j}{\mathbf{z}_1'^\top\mathbf{z}'_1}\mathbf{z}'_1$ for $2 \leq j \leq d$. Now $\mathbf{z}'_1$ and $\mathbf{z_j}$ are orthogonal for $2 \leq j \leq d$.

Otherwise the vectors $\mathbf{z}_j$ are self-orthogonal for $1 \leq j \leq d$, and $\mathbf{z}_1 + \mathbf{z}_j$ is self-orthogonal for $2 \leq j \leq d$. The latter implies $\mathbf{z}_1^\top\mathbf{z}_1 + 2\mathbf{z}_1\mathbf{z}_j + \mathbf{z}_j^\top\mathbf{z}_j = 2\mathbf{z}_1^\top\mathbf{z}_j = 0$. Hence $\mathbf{z}_1^\top\mathbf{z}_j = 0$ in characteristic $\neq 2$. Set $\mathbf{z}'_1 = \mathbf{z}_1$.

In all three cases $\mathbf{z}'_1, \mathbf{z}_2, \dots, \mathbf{z}_d$ comprise a basis of $\mathcal{C}$ such that $\mathbf{z}'_1$ is orthogonal to the space generated by the remaining vectors $\mathbf{z}_2, \dots, \mathbf{z}_d$.

The result now follows by induction. ☐

Lemma 25. *The self-orthogonal vectors of an orthogonal basis of $\mathcal{C}$ form a basis for $\mathcal{C} \cap \mathcal{C}^\perp$.*

Proof. Let $\mathbf{z}_1, \dots, \mathbf{z}_d$ form an orthogonal basis for $\mathcal{C}$, and $\mathbf{z} = \sum_{1\leq j\leq d} a_j\mathbf{z}_j \in \mathcal{C} \cap \mathcal{C}^\perp$. For $1 \leq i \leq d$ we have $0 = \mathbf{z}^\top\mathbf{z}_i = \sum_{1\leq j\leq d} a_j\mathbf{z}_j^\top\mathbf{z}_i = a_i\mathbf{z}_i^\top\mathbf{z}_i$. Hence if $\mathbf{z}_i^\top\mathbf{z}_i \neq 0$ then $a_i = 0$. It follows that $\mathbf{z}$ is generated by the self-orthogonal vectors of the basis, which, being independent, therefore form a basis of $\mathcal{C} \cap \mathcal{C}^\perp$. ☐

Proposition 26. *Let $\mathcal{C}$ be a subspace of $\mathbb{F}_3^E$. Then*

$$\sum_{\mathbf{z}\in\mathcal{C}} \omega^{|\mathrm{supp}(\mathbf{z})|} = (-1)^{d+d_1}(i\sqrt{3})^{d+d_0},$$

where $d = \dim\mathcal{C}$, $d_0 = \dim(\mathcal{C}\cap\mathcal{C}^\perp)$, and d_1 is the number of basis vectors of support size congruent to 1 modulo 3 in any orthogonal basis of $\mathcal{C}$.

Proof. Observe that for $\mathbf{z} \in \mathbb{Z}_3^E$ we have $|\mathrm{supp}(\mathbf{z})| \equiv \mathbf{z}^\top\mathbf{z} \pmod 3$. It follows that $\omega^{|\mathrm{supp}(\mathbf{z})|} = \omega^{\mathbf{z}^\top\mathbf{z}}$.

By Lemma 24 there is an orthogonal basis $\{\mathbf{z}_1, \dots, \mathbf{z}_d\}$ of $\mathcal{C}$. In particular, the inner product of $\mathbf{z} =$

$\sum_{1\le j\le d} a_j\mathbf{z}_j$ with itself is equal to $\sum_{1\le j\le d} a_j^2\mathbf{z}_j^\top\mathbf{z}_j$. So we find that

$$\begin{aligned}\sum_{\mathbf{z}\in\mathcal{C}}\omega^{\mathbf{z}^\top\mathbf{z}} &= \sum_{(a_1,\dots,a_d)\in\mathbb{Z}_3^d}\omega^{\sum_{1\le j\le d}a_j^2\mathbf{z}_j^\top\mathbf{z}_j}\\ &= \sum_{(a_1,\dots,a_d)\in\mathbb{Z}_3^d}\prod_{1\le j\le d}\omega^{a_j^2\mathbf{z}_j^\top\mathbf{z}_j}\\ &= \prod_{1\le j\le d}\sum_{a_j\in\mathbb{Z}_3}\omega^{a_j^2\mathbf{z}_j^\top\mathbf{z}_j}\\ &= \prod_{1\le j\le d}(1+2\omega^{\mathbf{z}_j^\top\mathbf{z}_j})\\ &= 3^{d_0}(1+2\omega)^{d_1}(1+2\omega^2)^{d-d_0-d_1},\end{aligned}$$

where d_0 (resp. d_1) is the number of vectors $\mathbf{z}_j, 1\le j\le d$, such that $\mathbf{z}_j^\top\mathbf{z}_j=0$ (resp. $=1$). With $1+2\omega=i\sqrt{3}$, $1+2\omega^2=-i\sqrt{3}$, and $d_0=\dim(\mathcal{C}\cap\mathcal{C}^\perp)$ by Lemma 25, the statement of the proposition now follows. ☐

As Gioan and Las Vergnas [22] observe in their Corollary 2, it is not obvious that the parity of the number of vectors in an orthogonal basis for $\mathcal{C}$ with support size congruent to 1 modulo 3 is independent of the choice of basis, a fact implied by Proposition 26.

We reach another polynomial time computable evaluation of the Tutte polynomial (bases for finite-dimensional vector spaces being easy to find by Gaussian elimination, and Lemma 24 providing a polynomial time algorithm for constructing an orthogonal basis):

Theorem 27. *Let $G=(V,E)$ be a graph and $\omega=e^{2\pi i/3}$. We have*

$$T(G;\omega,\omega^2)=(-1)^{d_2}\omega^{|E|+d}(i\sqrt{3})^{d_0},$$

where d_0 is the dimension of the space of $\mathbb{Z}_3$-tension-flows of G, d the dimension of the space of $\mathbb{Z}_3$-flows, and d_2 is the number of vectors with support size congruent to 2 modulo 3 in any orthogonal basis for the space of $\mathbb{Z}_3$-flows.

Proof. Setting $k=3$ and $x=\omega^2=\omega^{-1}$ in equation (11) we have

$$\sum_{\mathbf{z}\in\mathcal{C}}\omega^{-|E|+|\mathrm{supp}(\mathbf{z})|}=(\omega^2-1)^dT(G;\omega^2,\omega),$$

where $d=\dim\mathcal{C}=n(G)$ is the dimension of the space of $\mathbb{Z}_3$-flows. Then by Proposition 26 and $\omega^2-1=i\sqrt{3}\omega$ we obtain

$$\omega^{-|E|}(-1)^{d+d_1}(i\sqrt{3})^{d+d_0}=(i\sqrt{3}\omega)^dT(G;\omega^2,\omega).$$

Since $T(G;\omega^2,\omega)$ is the complex conjugate of $T(G;\omega,\omega^2)$ the result follows. ☐

It is interesting to note that although $\mathbb{Z}_3$-tension-flows are self-dual (flow-tensions are tension-flows), which leads one to expect their number to be counted by some evaluation of $T(G;x,y)$ on the line $x=y$, here the point (ω,ω^2) lies on the line $x=\overline{y}$ in the complex plane.

In Section 3.7.1 we saw that $T(G;-1,-1)=(-1)^{|E(G)|}(-2)^{b(G)}$, where $b(G)$ is the bicycle dimension of G, i.e., the dimension of the the subspace of $\mathbb{Z}_2$-tension-flows. The point $(-1,-1)$ lies on the hyperbola $(x-1)(y-1)=4$, so that by identity (11)

$$T(G;-1,-1)=(-2)^{-n(G)}\sum_{\mathbb{Z}_2\times\mathbb{Z}_2\text{-flows }\mathbf{z}}(-1)^{|E|-|\mathrm{supp}(\mathbf{z})|}.$$

This might lead one to expect rather an expression for $T(G;-1,-1)$ in terms of the space of $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows in $\mathbb{F}_4^E$. Indeed, the dimension of the space of $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows is equal to the bicycle dimension $b(G)$. A $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flow decomposes by projection into a pair of $\mathbb{Z}_2$-tension-flows, and conversely such a pair of $\mathbb{Z}_2$-tension-flows can be pieced together to make a $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flow. Hence there are precisely $(2^{b(G)})^2$ vectors that are $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows, i.e., they comprise a space of dimension $b(G)$ over $\mathbb{F}_4$. Hence we could also have written that $T(G;-1,-1)=(-1)^{|E|}(-2)^{d_0}$, where d_0 is the dimension of the space of $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows.

Question 14 Are there in general as many $\mathbb{Z}_4$-tension-flows as $\mathbb{Z}_2 \times \mathbb{Z}_2$-tension-flows?

Vertigan [50] proved that the Tutte polynomial evaluated at the point $(i, -i)$ on the hyperbola $(x-1)(y-1) = 2$ has the following interpretation:

Theorem 28 ([50])**.** *Let G be a graph with bicycle dimension $b(G)$. Then*

$$|T(G; i, -i)| = \begin{cases} \sqrt{2}^{b(G)} & \text{if every bicycle has size a multiple of 4,} \\ 0 & \text{otherwise.} \end{cases}$$

For example, $T(C_4; i, -i) = i^3 + i^2 + i - i = -i - 1 = -\sqrt{2}\frac{1+i}{\sqrt{2}}$, where $\frac{1+i}{\sqrt{2}}$ is a primitive eighth root of unity. Recall also that every bicycle has even size, so that the bicycles of size a multiple of 4 either comprise all bicycles, or exactly half of them. Theorem 28 implies a polynomial time algorithm for evaluating $T(G; i, -i)$. [What is the argument of $T(G; i, -i)$ as a complex number?]

3.8 The beta invariant

In this chapter we have been mainly concerned with evaluations of the Tutte polynomial. In this section we concentrate on one of its coefficients.

The coefficient $t_{1,0}(G)$ is known as Crapo's *beta invariant*, or also the *chromatic invariant*, with $t_{1,0}(G) = (-1)^{|V(G)|}P'(G; 1)$. By Propositions 13 and 12, $t_{1,0}(G) = t_{1,0}(G^*)$ when G is a connected planar graph.

We know from the corresponding property of the chromatic polynomial that the beta invariant is unaffected by the addition or removal of parallel edges. A direct proof can be given by a deletion/contraction of a parallel edge, noting that $t_{1,0}(G) = 0$ if G has a loop. Also, since $t_{1,0}(G) = t_{0,1}(G)$ when G has at least two edges, we can use the property of the flow polynomial that is invariant under edge subdivisions (the number of nowhere-zero $\mathbb{Z}_k$-flows being unaffected) to deduce that $t_{1,0}(G)$ is unaffected by subdivision of an edge (when G has at least two edges). We shall generalize these observations, and to do so we introduce the notions of series and parallel connections of graphs.

3.8.1 Series and parallel connections

Suppose $G_i = (V_i, E_i)$, $i = 1, 2$, are vertex-disjoint graphs, and that we are given edges $e_1 = u_1v_1 \in E_1$ and $e_2 = u_2v_2 \in E_2$. The *parallel connection* of G_1 and G_2 along edges e_1 and e_2 is formed by first deleting edges e_1 and e_2, identifying u_1 with u_2 to make a new vertex u, identifying v_1 with v_2 to make a new vertex v, and finally joining u and v with a new edge e. (See Figure 4.) The parallel connection as defined has vertex set $V_1 \cup V_2 \cup \{u, v\} \setminus \{u_1, v_1, u_2, v_2\}$ and edge set $E_1 \cup E_2 \cup \{e\} \setminus \{e_1, e_2\}$. The *series connection* of G_1 and G_2 along edges e_1 and e_2 is formed by first deleting edges e_1 and e_2, then identifying u_1 with u_2 to make a new vertex u, and finally joining vertices v_1 and v_2 by a new edge e. The series connection has vertex set $V_1 \cup V_2 \cup \{u\} \setminus \{u_1, u_2\}$ and edge set $E_1 \cup E_2 \cup \{e\} \setminus \{e_1, e_2\}$.

Strictly speaking we ought to choose *oriented* edges e_1 and e_2, but in fact it will not matter what order we choose the endpoints u_i and v_i of e_i in making series and parallel connections, since up to 2-isomorphism the result will be the same. Since we are interested in the behaviour of the Tutte polynomial and this is a 2-isomorphism invariant, we can leave the choice arbitrary.

Let $\mathcal{C}(G)$ denote the set of circuits of a graph G. Then the set of circuits of the series connection of G_1 and G_2 is given by

$$\mathcal{C}(G_1 \setminus e_1) \cup \mathcal{C}(G_2 \setminus e_2) \cup \{(C_1 \setminus e_1) \cup (C_2 \setminus e_2) \cup \{e\} : e_i \in C_i \in \mathcal{C}(G_i), i = 1, 2\}.$$

The set of circuits of the parallel connection of G_1 and G_2 is likewise given by

$$\mathcal{C}(G_1 \backslash e_1) \cup \{(C_1 \setminus e_1) \cup \{e\} : e_1 \in C_1 \in \mathcal{C}(G_1)\}$$

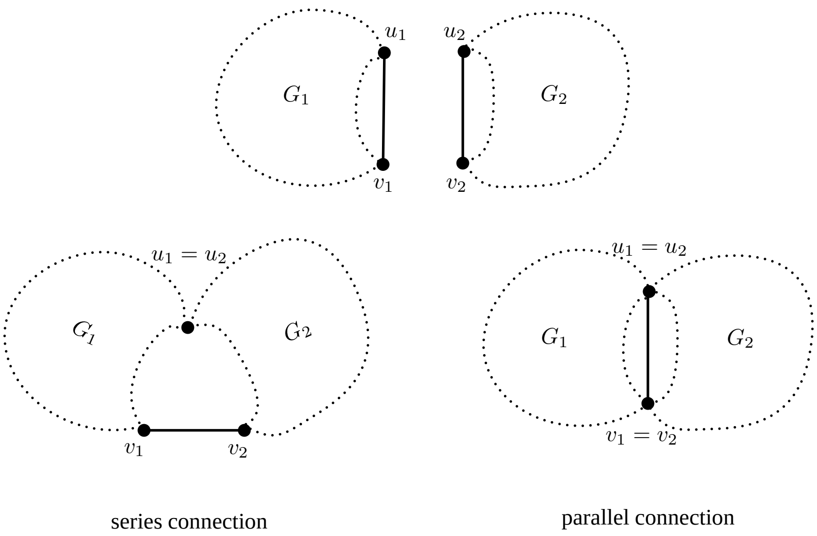

Figure 4: Series and parallel connection of vertex-disjoint graphs G_1 and G_2 along edges u_1v_1 and u_2v_2, both of which are deleted before making the vertex identification(s) involved in either operation.

$$\cup\, \mathcal{C}(G_2 \backslash e_2) \cup \{(C_2 \setminus e_2) \cup \{e\} : e_2 \in C_2 \in \mathcal{C}(G_2)\}$$
$$\cup\, \{(C_1 \backslash e_1) \cup (C_2 \backslash e_2) : e_i \in C_i \in \mathcal{C}(G_i), i = 1, 2\}.$$

Question 15

(i) Suppose $e_1 = u_1v_1$ is a loop in G_1, i.e., $u_1 = v_2$. Show that the parallel connection of G_1 and G_2 along edges e_1 and e_2 is 2-ismorphic to he disjoint union of G_1 and G_2/e_2. (In terms of matroids, the cycle matroid of the parallel connection is isomorphic to the 2-sum of the cycle matroids for G_1 and G_2/e_2.)

(ii) Suppose $e_1 = u_1v_2$ is a bridge in G_1. Show that the series connection of G_1 and G_2 along edges e_1 and e_2 is 2-isomorphic to the disjoin union of G_1 and $G_2 \backslash e_2$.

We observe that the edge e formed by the parallel connection of G_1 and G_2 along edges e_1 and e_2 is a loop if and only if either e_1 or e_2 is a loop. Likewise, the edge e formed in the series connection of G_1 and G_2 is a bridge if and only if either e_1 or e_2 is a bridge.

Definition 29. *A graph is* series-parallel *if it can be constructed from K_2 by a sequence of the following two operations:*

(i) subdividing an edge (introducing a vertex of degree 2),

(ii) placing an edge parallel to an existing edge.

A 2-connected series-parallel graph can be constructed from C_2 (a pair of vertices joined by two parallel edges) by a sequence of series and parallel extensions. Series-parallel graphs are loopless and planar. See Figure 5 to see how series-parallel extensions are particular cases of series-parallel connections.

Proposition 30. *Let $G_i = (V_i, E_i)$, $i = 1, 2$, be graphs.*

(i) Suppose G is the graph obtained by joining G_1 and G_2 in series along edges e_1 and e_2, at least one of them not a bridge. Then T forms a spanning tree of G if and only if $T \cap E_i$ is a spanning tree of G_i for $i = 1, 2$.

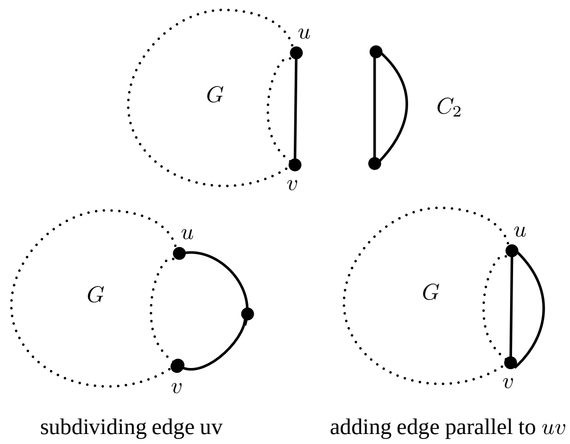

Figure 5: Series-parallel extensions are series-parallel connections with C_2.

(ii) Suppose now G is obtained by joining G_1 and G_2 in parallel along edges e_1 and e_2, at least one of them not a loop. Then T is a spanning tree of G containing the edge e formed by the connection if and only if $T \cap E_i$ is a spanning tree of G_i containing e_i for $i = 1, 2$. Moreover, T is a spanning tree of G not containing e if and only if $e \notin T$ and the sets $(T \cap E_i) \cup \{e_i\}$ and $T \cap E_j$ are spanning trees of G_i and G_j respectively, where $\{i, j\} = \{1, 2\}$.

Lemma 31. *If G is 2-connected and G/e is not, then G is a parallel connection. If G is 2-connected and $G\backslash e$ is not, then G is a series connection.*

See [45, Theorem 7.1.16].

3.8.2 Properties of the beta invariant

Lemma 32. *The beta invariant $t_{1,0}(G)$ is multiplicative over series and parallel connections.*

Proof. By Proposition 30 we can analyse spanning tree activities of a series or parallel connection of graphs G_1 and G_2 in terms of those of G_2 and G_2. By Tutte's interpretation of $t_{1,0}(G)$ as the number of spanning trees of internal activity 1 and external activity 0 [...] □

Theorem 33. *Let G be a loopless 2-connected graph. Then $t_{1,0}(G) \geq 1$ with equality if and only if G is series-parallel.*

Proof. If G is not 2-connected then $t_{1,0}(G) = 0$.

We prove the statement by induction on the number of edges. The base case C_2 has $T(C_2; x, y) = x + y$.

Suppose G is 2-connected with $m \geq 3$ edges and assume the truth of the assertion for 2-connected graphs with less than m edges. If G has an edge e that has been introduced in series (one of its endpoints has degree 2), then G/e is 2-connected while $G\backslash e$ is not. Hence $t_{1,0}(G\backslash e) = 0$ while by inductive hypothesis $t_{1,0}(G/e) = 1$

On the other hand, if e is parallel to another edge of G then G/e has a loop and at least one other edge and hence is not 2-connected, while $G\backslash e$ is 2-connected. By inductive hypothesis we have $t_{1,0}(G\backslash e) = 1$, so that $t_{1,0}(G) = 0 + t_{1,0}(G) = 1$.

For the converse, that $t_{1,0}(G) = 1$ implies G is series-parallel, we again proceed by induction on the number of edges. Since $t_{1,0}(G) = 1$, G is 2-connected. Take e any edge. Since $t_{1,0}(G) = t_{1,0}(G\backslash e) + t_{1,0}(G/e)$, exactly one of G/e or $G\backslash e$ is 2-connected (and incidentally the other is series-parallel by the inductive hypothesis, but this is not needed).

Consider the case where G is 2-connected but G/e is not. Hence G is a parallel connection, and by multiplicativity of $t_{1,0}(G)$ over series-parallel connections it has to be a parallel connection of series-parallel graphs. But this is again series-parallel.

The case where $G\backslash e$ is similar. ☐

Proofs given of Theorem 33 usually invoke Dirac's characterization of series-parallel graphs as precisely those with no K_4 minor [20] and a fact first proved by Brylawski [18] that if G is 2-connected and H is a (non-empty) minor of G then $t_{i,j}(H) \leq t_{i,j}(G)$. The proof we have presented follows Zavlasky [56].

Question 16

(i) Let $W_n = K_1 + C_n$ be the wheel on $n+1$ vertices (an n-cycle all of whose vertices are joined to a new central vertex). In the chapter on the chroamtic polynomial you calculated the chromatic polynomial of W_n: leaning on your past labours, deduce the value $t_{1,0}(W_n)$. If it had slipped your mind to find $P(W_n; z)$ earlier, then what alternative method might you have used to calculate $t_{1,0}(W_n)$?

(ii) By using $P(K_n; z) = z^{\underline{n}}$, show that $t_{1,0}(K_n) = (n-2)!$.

Proposition 34. *If $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = s \geq 2$ and the induced subgraph on $V(G_1) \cap V(G_2)$ is a clique K_s, then*

$$t_{1,0}(G) = t_{1,0}(G_1) t_{1,0}(G_2)/(s-2)!.$$

Note that if G has a cut-vertex ($s = 1$) then $t_{1,0}(G) = 0$.

Proof. This follows from the expression for the chromatic polynomial of a quasi-separation (see Prop. 8 in the chapter on chromatic polynomial) written as

$$P(G; 1-z) P(K_s; 1-z) = P(G_1; 1-z) P(G_2; 1-z),$$

where, for connected G,

$$P(G; 1-z) = (1-z) \sum_{1 \leq i \leq |V|-1} (-1)^{|V|-1-i} t_{i,0}(G) z^i,$$

and the fact that $t_{1,0}(K_s) = (s-2)!$. Comparing coefficients of z^2 gives the result. ☐

In particular, edge-glueing a series-parallel graph to G does not change its beta invariant.

The only 3-connected graph G with beta invariant $t_{1,0}(G) = 2$ is K_4, and a similar classification of 3-connected graphs with beta invariant up to 9 has been made (see references given in [44, §7.1]). An *outerplanar* graph is a planar graph with an embedding in the plane with the property that all vertices of G lie on the outer face. A graph is outerplanar if and only if it has no K_4 minor (so it is series-parallel) or $K_{2,3}$ minor.

Theorem 35. [24] *If G is a simple 2-connected series-parallel graph then $t_{2,0}(G) \geq t_{0,2}(G) + 1$ with equality if and only if G is outerplanar.*

It turns out that the beta invariant $t_{1,0}(G)$ counts a certain subset of those acyclic orientations counted by $T(G; 1, 0)$ (Theorem 7 above).

Theorem 36. [Greene and Zaslavsky, 1983; Las Vergnas, 1984][1] *Let G be a connected graph and $uv \in E(G)$. The number of acyclic orientations of G with u as unique source and v as unique sink is equal to $t_{1,0}(G)$.*

[1] The original proofs of Greene and Zaslavsky of this result and Theorem 7 use hyperplane arrangements. A contraction–deletion proof was given by Gebhard and Sagan [21]. Las Vergnas proved a stronger theorem in [35], giving an orientation expansion for the Tutte polynomial.

Proof. Let $Q_{uv}(G)$ denote the number of acyclic orientations of G with u as unique source and v as unique sink.

Recall that $t_{1,0}(G) = 0$ if G is not 2-connected. We know that $t_{1,0}(G) = t_{1,0}(G/e) + t_{1,0}(G\backslash e)$ for an ordinary edge e, and if G has more than one edge and e is a bridge of G then $t_{1,0}(G) = 0$ (since G is not 2-connected). Also $t_{1,0}(K_2) = 1$. Finally, $t_{1,0}(G) = 0$ if G has a loop e.

When G is not 2-connected it is impossible to have an acyclic orientation of G with unique source u and unique sink v. First, if G is not connected then there are not even any acyclic orientations with unique source u, since each component has a source. Second, if G is connected and $G \cong G_1 \cup G_2$ with $|V(G_1) \cap V(G_2)| = 1$, then an acyclic orientation restricted to G_1 has at least one source and sink, at least one of which survives as a source or sink in G. Similarly for G_2. But then there is either a source or sink in G_1 and in G_2, and these are not connected by an edge. Hence u and v are not unique as source and sink.

Clearly $Q_{uv}(K_2) = 1$ and $Q_{uv}(G) = 0$ if G has a loop.

If G has at least two edges, is 2-connected and has no loops, then G has no bridges. It remains to prove that in this case $Q_{uv}(G) = Q_{uv}(G/e) + Q_{uv}(G\backslash e)$, where e is an ordinary edge. We can choose $e = wv$ with $w \neq u, v$. In an acyclic orientation of G with unique sink v the edge wv is directed from w to v. Since u is the unique source there is at least one edge directed into w. If there is also at least one other edge directed out of w, then deleting e gives an acyclic orientation of $G\backslash e$ with unique source u and unique sink v. On the other hand, if e is the only edge directed out of w then contracting the edge e gives an acyclic orientation of G/e with unique source u and unique sink v (which is identified with w in the graph G/e). Thus partitioning acyclic orientations of G with unique source u and unique sink v according to whether or not $G\backslash wv$ is also an acyclic orientation with this property, we find that $Q_{uv}(G) = Q_{uv}(G/wv) + Q_{uv}(G\backslash wv)$. □

Question 17 Formulate and prove the dual statement to Theorem 36 that concerns totally cyclic orientations. (Since $t_{1,0}(G) = t_{0,1}(G)$ when G has at least two edges, the number of this type of totally cyclic orientation turns out to be the same as the number of acyclic orientations with prescribed source and sink.)

3.9 Computational complexity

We have seen that the Tutte polynomial can be computed in polynomial time at some particular points. Specifically, these points are: $(0, 0)$ (whether there are any edges), $(1, 1)$ (number of spanning trees), $(2, 2)$ (number of subgraphs), $(-1, 0)$ (whether bipartite or not), $(0, -1)$ (whether Eulerian or not), $(-1, -1)$ (up to easily determined sign equal to number of bicycles), and also in the last section interpretations for evaluations at $(e^{2\pi i/3}, e^{-2\pi i/3})$ and $(i, -i)$, the former involving the dimension of the space spanned by vectors that are simultaneously $\mathbb{Z}_3$-flows and $\mathbb{Z}_3$-tensions.

Recall also that $T(G; x, y) = (x-1)^{r(G)} y^{|E(G)|}$ when $(x-1)(y-1) = 1$, so that the Tutte polynomial is also polynomial time computable at points on this hyperbola (the points $(0, 0)$ and $(2, 2)$ were already mentioned in the previous paragraph).

Theorem 37 below says that we have in fact now encountered all such "easy points".

A computational (enumeration) problem can be regarded as a function mapping inputs to solutions (graphs to the number of their proper vertex 3-colourings, for example). A problem is *polynomial time computable* if there is an algorithm which computes the output in length of time (number of steps) bounded by a polynomial in the size of the problem instance. The class of such problems is denoted by P. If A and B are two problems, we say that A is *polynomial time reducible* to B, written $A \propto B$, if it is possible with the help of a subroutine for problem B to solve problem A is polynomial time.

The class $\#\mathsf{P}$ can be roughly described as the class of all enumeration problems in which the structures being counted can be recognized in polynomial time (i.e., instances of an NP problem). For example, counting Hamiltonian paths in a graph is in $\#\mathsf{P}$ because it is easy to check whether a given set of edges is a Hamitonian path.

The class $\#\mathsf{P}$ has a class of "hardest" problems called the $\#\mathsf{P}$-complete problems. A problem A belonging to $\#\mathsf{P}$ is $\#\mathsf{P}$-complete if for any other problem B in $\#\mathsf{P}$ we have $B \propto A$. A prototypical example of a $\#\mathsf{P}$-complete problem is #SAT, the problem of counting the number of satisfying assignments of a Boolean function. Many of the thousands of problems known to be #P complete have been shown to be so by reduction to #SAT. Counting Hamiltonian paths is an example of a #P-complete problem (even when restricted to planar graphs with maximum degree 3).

A problem is #P-hard if any problem in #P is polynomial time reducible to it. In other words, A is #P-hard if the existence of a polynomial time algorithm for A would imply the existence of a polynomial time algorithm for any problem in #P. (A #P-hard problem is #P-complete if it belongs to the class #P itself.)

Many evaluations of the Tutte polynomial count structures associated with a graph. Sometimes though it is not apparent what an evaluation of the Tutte polynomial at a particular point (a, b) might count. However, we can still speak of whether the problem of computing $T(G; a, b)$ can be done in polynomial time or if it is a #P-hard problem (being able to evaluate it for any graph in polynomial time would imply that every problem in #P could be computed in polynomial time).

Theorem 37 ([29]). *Evaluating the Tutte polynomial of a graph at a particular point of the complex plane is #P-hard except when either*

(i) the point lies on the hyperbola $(x-1)(y-1) = 1$,

(ii) the point is one of the special points $(1, 1)$, $(-1, 0)$, $(0, -1)$, $(-1, -1)$, $(i, -i)$, $(-i, i)$, $(e^{2\pi i/3}, e^{-2\pi i/3})$, $(e^{-2\pi i/3}, e^{2\pi i/3})$.

In the special cases (i) and (ii) evaluation can be carried out in polynomial time.

In [49] Vertigan and Welsh show that the same statement in Theorem 37 holds even when restricting the problem to computing the Tutte polynomial for bipartite graphs.

Around the same time as [49], but only much later published, Vertigan showed that restricting the problem of evaluating the Tutte polynomial to planar graphs only yields extra "easy points" on the hyperbola $(x-1)(y-1) = 2$ (corresponding to the partition function of the Ising model, which in the planar case is polynomial time computable due to Kasteleyn's expression for the partition function of the Ising model as the Pfaffian of an associated matrix).

Theorem 38 ([52]). *The problem of computing the Tutte polynomial of a planar graph at a particular point of the complex plane is #P-hard except when either*

(i) the point lies on one the hyperbolae $(x-1)(y-1) = 1$ *or* $(x-1)(y-1) = 2$,

(ii) the point is one of the special points $(1, 1)$, $(-1, 0)$, $(0, -1)$, $(-1, -1)$, $(e^{2\pi i/3}, e^{-2\pi i/3})$, $(e^{-2\pi i/3}, e^{2\pi i/3})$.

In the special cases (i) and (ii) evaluation can be carried out in polynomial time.

See e.g. [53] for a more detailed account of the complexity of counting problems, with special emphasis on those related to the Tutte polynomial.

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