The Tutte polynomial and related polynomials

Andrew Goodall

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1 Preliminary remarks

The Tutte polynomial was defined originally for graphs and extends to more generally to matroids. The many natural combinatorial interpretations of its evaluations and coefficients for graphs then translate to not obviously related combinatorial quantities in other matroids. For example, the Tutte polynomial evaluated at $(2, 0)$ gives the number of acyclic orientations of a graph.[1] Zaslavsky proved that the Tutte polynomial at $(2, 0)$ also counts the number of different arrangments of sets of hyperplanes in n-dimensional Euclidean space (the underlying matroid is of a set of n-dimensional vectors over $\mathbb{R}$).[2]

For another example of extending the scope of a definition, the Tutte polynomial $T(G; x, y)$ along the hyperbola $xy = 1$ when G is planar specializes to the *Jones polynomial* of the alternating link associated with G (via the medial graph of G). Starting from the knot theory context, analogues of the Tutte polynomial have recently been defined for signed graphs (needed for encoding arbitrary links, not just alternating) and embedded graphs (in other surfaces than the plane).

We shall see the connection of the Tutte polynomial and the Jones polynomial via the *Kauffman bracket* of a link: the deletion-contraction recurrence for the Tutte polynomial of a graph is mirrored in the skein relation for the Jones polynomial (involving local transformations of a knot).

But before this we shall see with the *interlace polynomial* another example of this phenomenon – originally defined meaningfully only for a restricted class of graphs (namely interlace graphs, or *circle graphs*), its recursive definition (analogous to deletion-contraction for the Tutte polynomial) applies to any graph. Interpreting its values for graphs generally remains an open area of research. Its definition has already been extended to matroids too.

[1] This amid the world of graphs is in itself surprising, as it is the evaluation of the chromatic polynomial at -1: this is the tip of the iceberg with regard to the connection between orientations and colourings of graphs

[2] A hyperplane in n-dimensional Euclidean space is an $(n-1)$-dimensional flat subset (congruent to $(n-1)$-dimensional space), i.e., affine subspace of dimension $n-1$. Flats in Euclidean spaces are affine subspaces such as points, lines, planes,... More generally, a *flat* in a matroid is a subset with the property that adding any other element to it increases its rank, and a *hyperplane* in a matroid of rank r is a flat of rank $r-1$.

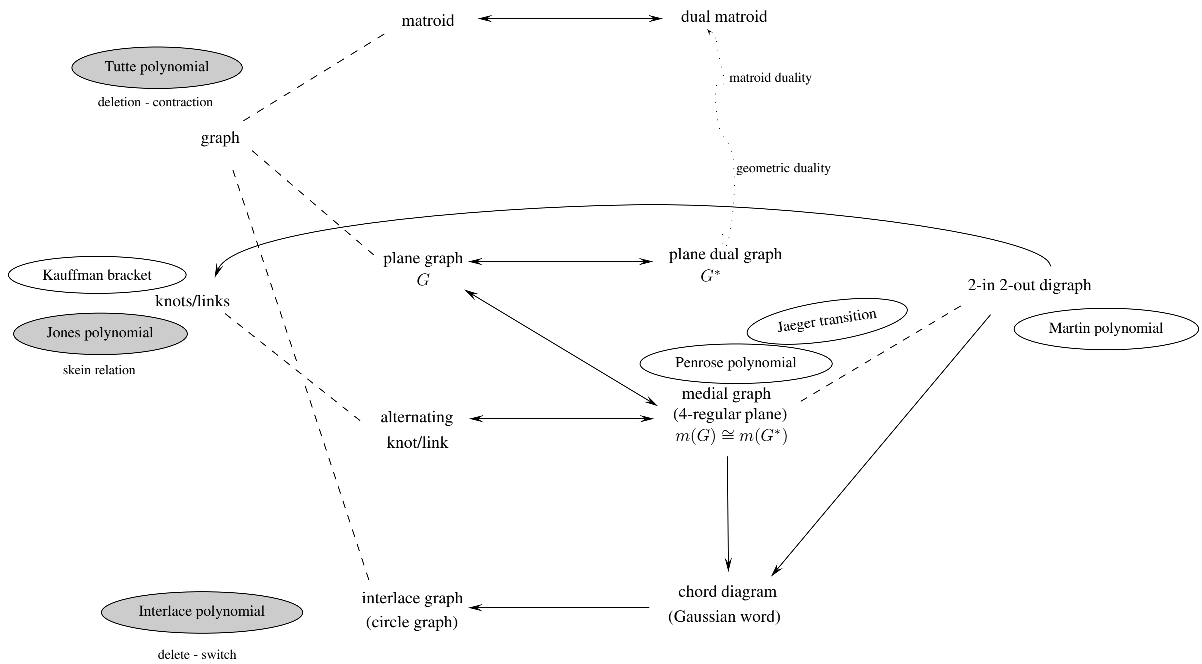


Figure 1: Schematic diagram of interrelationships between various combinatorially defined polynomials. The dashed lines from one class up to another indicate containment. Arrows indicate operations (invertible when double arrow) from one class to another. Once defined for a class of objects, a polynomial may be extended in its scope by lifting up to a more general class of objects – this is what happens for the Tutte polynomial (graphs to matroids) and interlace polynomial (circle graphs to graphs to binary matroids). In the reverse direction, restricting the Jones polynomial to alternating links gives the Tutte polynomial along $xy = 1$. Missing from the diagram are the class of signed plane graphs: these encode all knots/links and by extending to signed graphs generally there is a *signed Tutte polynomial* which satisfies a deletion-contraction recurrence, only with two cases according to the sign of the edge.

Common to the application to knots and to Eulerian tours on 2-in 2-out digraphs is the operation of taking the medial graph of a plane graph and considering the types of transition that may occur at a vertex when following the knot (under or over) or Eulerian tour of the graph (which of the edges to follow out of the vertex). There are three possible types of transition when following edges in and out of at a vertex of degree 4, and this would bring us to Jaeger's *transition polynomial* (see [2]) which includes the *Penrose polynomial* as a special case, and is also intimately related to the Tutte polynomial. However, we shall only have time to consider the interlace polynomial and Jones polynomial.

2 Graphs, duality and matroids

2.1 Cuts, circuits and cycles

We start with undirected graphs. A set A of edges in a graph $G = (V, E)$ is an *edge cut*, or *cutset*, if the graph $G - A = (V, E \setminus A)$ has more components than G. In most cases we shall be able to assume that G is a connected graph. In this case a cutset is a set of edges which disconnects the graph. An inclusion minimal edge cut of a graph G is called a *bond*. It is easy to see that a bond is always contained in a single component of G. If A is a bond of a connected graph G then the graph $G - A$ has exactly two components, say with vertex sets V_1 and V_2. A is the the set of all edges between V_1 and V_2, denoted by $E(V_1, V_2)$. A cut of size one is called a *bridge*.

A *path* we interpret both as a subgraph and as a sequence $v_0, e_1, v_1, \ldots, e_t, v_t$ of vertices and edges of the graph, in which the vertices v_i (and hence edges too) are distinct. In a *trail* vertices may be repeated, only edges e_i being distinct, and in a *walk* both vertices and edges may be repeated. A *circuit* of length t is formed by a sequence $v_0, e_1, v_1, \ldots, e_t, v_t$ in which all vertices are distinct with the exception only of $v_0 = v_t$ (so it can be thought of as a “closed path”) . Similarly, a trail $v_0, e_1, v_1, \ldots, e_t, v_t$ satisfying $v_0 = v_t$ is called a *closed trail* or a *cycle*.[3]

As all the edges of a closed trail are distinct, a closed trail may be identified with the subset of edges traversed by it. By saying that a set of edges is a cycle is meant that for some ordering it will be form a closed trail. If all the edges of the connected graph G are traced by some cycle then G is an *Eulerian graph*.

Proposition 2.1 *A graph is Eulerian if and only if it is connected and all its vertices have even degree.*

It follows that any Eulerian subgraph is an edge-disjoint union of circuits (this is sometimes called Veblen’s theorem [14]). The following are two algorithms for constructing an Eulerian cycle of a connected graph G all of whose vertices are of even degree:

(i) **Fleury’s algorithm** [6] (“Don’t burn your bridges”) starts with an arbitrarily chosen vertex v_0 and an arbitrary starting edge e_0 incident with v_0. The edge e_0 is deleted and its other endpoint is the next vertex v_1 to be chosen. At each stage, at current vertex v_i an edge e_i can be chosen as the next edge in the cycle if it is not a bridge in the remaining graph, unless there is no such edge, in which case the only remaining edge left at the current vertex is chosen. It then moves to the other endpoint v_{i+1} of edge e_i, after which e_i is deleted from the graph. At the end of the algorithm there are no edges left, and the sequence in which the edges were chosen forms an Eulerian cycle. [In this algorithm, the last edges chosen from v_i ($i > 0$) before returning to v_0 form a spanning tree of G – each of them is a bridge by definition of the algorithm.]

[3] Circuits and cycles have been defined this way round in order to have circuits in the graph theory sense coincide with circuits in the matroid theory sense. For us any circuit is a cycle, not vice versa. However, this is sometimes counter to graph theory terminology found elsewhere, where cycles are circuits....

(ii) **Hierholzer's algorithm** [9] chooses any starting vertex v_0, and follows a trail of edges from that vertex until it returns to v_0. It is not possible to get stuck at any vertex other than v_0, because all vertex degrees being even ensures that when the trail enters a vertex $v \neq v_0$ there must be an unused edge leaving v. The trail formed in this way is closed, but may not cover all the vertices and edges of the initial graph. As long as there exists a vertex v belonging to the current cycle that has incident edges not yet in the cycle, start another trail from v, following unused edges until returning to v, and join the cycle formed in this way to the previous cycle. [This algorithm may be thought of as gluing cycles together to form a Eulerian cycle covering all the edges of G, cf. Veblen's theorem that an Eulerian cycle can be partitioned into circuits.]

Proposition 2.2 *The intersection of any cutset with any cycle is even.*

Question 1

(i) Prove (or recall the proofs of) Propositions 2.1 and 2.2.

(ii) Prove that the symmetric difference of two bonds contains a bond.

(iii) Prove that the symmetric difference of two circuits contains a circuit.

2.2 Orthogonality of cycles and cutsets

If $\mathbb{F}$ is a field then the set of all vectors of length m with entries in $\mathbb{F}$ forms a vector space $\mathbb{F}^m$ of dimension m. This vector space is equipped with inner product $\mathbf{x}\mathbf{y}^\top = \sum_{i=1}^m x_i y_i$. If V is a vector subspace of $\mathbb{F}^m$ then the set of all vectors $\mathbf{y}$ which are orthogonal to all vectors $\mathbf{x} \in V$, is again a vector space which is called the *orthogonal complement* of V and denoted by $V^\perp$. Observe that $V^{\perp\perp} = V$ and that $\dim V^\perp = m - \dim(V)$.

Consider a graph $G = (V, E)$ with m edges and the field $\mathbb{F} = \mathbb{Z}_2$ on two elements. The vector space $\mathbb{Z}_2^E$ formed by all vectors $\mathbf{x} = (x_e : e \in E)$, where $x_e \in \mathbb{Z}_2$, is isomorphic to the vector space $\mathbb{Z}_2^m$. A vector $\mathbf{x}$ may be thought of as the characteristic (indicator) vector of a subset of edges of G. Now consider the set $\mathcal{E}$ of all vectors $\mathbf{x}$ which correspond to cycles (i.e., to Eulerian subgraphs). Also, denote by $\mathcal{C}$ the set of all vectors $\mathbf{x}$ which correspond to edge cuts of G. We have the following basic result:

Theorem 2.3 *For any graph G the following hold:*

(i) Both $\mathcal{E}$ and $\mathcal{C}$ are vector subspaces of $\mathbb{Z}_2^E$.

(ii) $\mathcal{E}$ and $\mathcal{C}$ are orthogonal complements of each other.

(iii) $\dim(\mathcal{E}) = m - n + k$, where $n = |V(G)|$ and k is the number of components of G, and $\dim(\mathcal{C}) = n - k$.

Before proving this result, recall that a *spanning forest* of a graph $G = (V, E)$ is any (edge-) inclusion maximal subgraph (V, A) not containing any cycle. A spanning forest is just a *spanning tree* if G is connected. Of course $|A| = |V| - c(G)$, where $c(G)$ is the number of components of G. A spanning forest (V, A) can be dually defined as an inclusion-minimal subgraph of G which has the same number of connected components as G.

PROOF. Both $\mathcal{E}$ and $\mathcal{C}$ are vector subspaces of $\mathbb{Z}_2^E$ by virtue of Question 1 above. Proposition 2.2 says that $\mathcal{C} \subseteq \mathcal{E}^\perp$, so that $\dim \mathcal{C} \leq m - \dim \mathcal{E}$. It therefore suffices to prove $\dim(\mathcal{E}) \geq m - n + k$ and $\dim(\mathcal{C}) \geq n - k$.

Let A be the edge set of a spanning forest of G (so $|A| = n-k$, $|E \setminus A| = m-n+k$). By maximality of A the graph $(V, A \cup \{e\})$ contains, for every $e \in E \setminus A$, a uniquely determined cycle Z_e containing the edge e. By the minimal connectivity definition of a spanning forest we know that for every $e \in A$ the graph $(V, A \setminus \{e\})$ has more components than G and thus there exists a unique cutset C_e of G containing the edge e with the same components as the graph $(V, A \setminus \{e\})$. But now both cycle Z_e and cut C_e are the only selected cycles and cuts containing the edge e Since $e \notin Z_f$ for each $f \in E \setminus (A \cup \{e\})$, and similarly $e \notin C_f$ for each $f \in A \setminus \{e\}$, the cycles $\{Z_e : e \in E \setminus A\}$ are linearly independent, as are the cutsets $\{C_e : e \in A\}$. This proves both $\dim(\mathcal{E}) \geq m - n + k$ and $\dim(\mathcal{C}) \geq n - k$. $\square$

Question 2

(i) Prove that if Z is a cycle in G and A the set of edges of a spanning forest of G then $\sum_{e \in Z \setminus A} Z_e = Z$ (for simplicity we identify in this notation a cycle with its characteristic vector).

(ii) Prove a similar statement for cuts.

2.3 Graph duality

Henceforth we allow graphs with parallel edges and loops. A *simple graph* is a graph with no parallel edges or loops. When clarity demands it, the term *multigraph* is used for a graph in which there may be parallel edges and loops.

Two simple graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijection $f : V \to V'$ such that $uv \in E$ if and only if $f(u)f(v) \in E'$, for all $u, v \in V$.

Definition 2.4 *Two multigraphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there are functions $f : V \to V'$ and $f : E \to E'$ such that*

(i) if e has endpoints u and v then $g(e)$ has endpoints $f(u)$ and $f(v)$;

(ii) f and g are bijections.

An isomorphism between multigraphs is an isomorphism between their underlying simple graphs together with the condition that edge multiplicities are the same (including

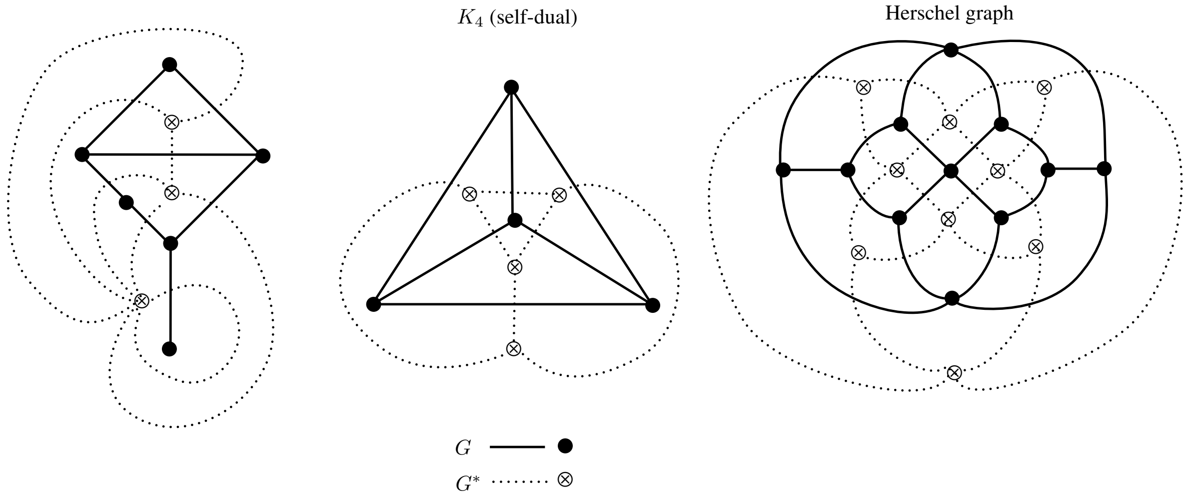


Figure 2: Some plane graphs and their duals. The leftmost example illustrates how edges in series become parallel edges in the dual and bridges become loops in the dual.

loops). A multigraph $G = (V, E)$ can be represented by its adjacency matrix $A = A(G)$ with (u, v) entry equal to the number of edges joining u and v. Multigraphs G and G' are isomorphic if and only if the matrices $A(G)$ and $A(G')$ are permutation-equivalent.

A graph $G = (V, E)$ is *planar* if it can be drawn in the plane so that in the drawing distinct arcs are openly disjoint and share only end-vertices in the case that corresponding edges are incident. A graph with such a drawing is called a *plane graph.* An edge may lie on the boundary of one face (and this if and only if it is a bridge) or two faces. As we are considering multigraphs, a face may be formed by only two edges.

Let $G = (V, E, F)$ be an (undirected) plane graph with set of faces F. The *geometric dual* of G is the graph $G^* = (V^*, E^*, F^*)$, with $V^* = F, E^* = \{e^* : e \in E\}$, where e^* has endpoints the faces of G which have e on their boundary (thus e^* is a loop when e is a bridge). The face set F^* of G^* may be identified with the vertex set of G; and then the edge set E^* may be identified with the edge set E of G.

The dual graph G^* is again planar and $(G^*)^* \cong G$.

A simple graph may have a graph with parallel edges and loops as its dual. For example, if T is a tree with n vertices then the dual of any of its plane drawings is a single vertex graph with $n-1$ loops. As another example consider any 2-connected plane graph G (so in particular there are no bridges) and let G' be the graph which arises from G by subdividing every edge of G by one vertex. Then the dual graph G'^* arises from G^* by replacing every edge by two "parallel" edges.

Further examples of dual pairs are shown in Figure 2.

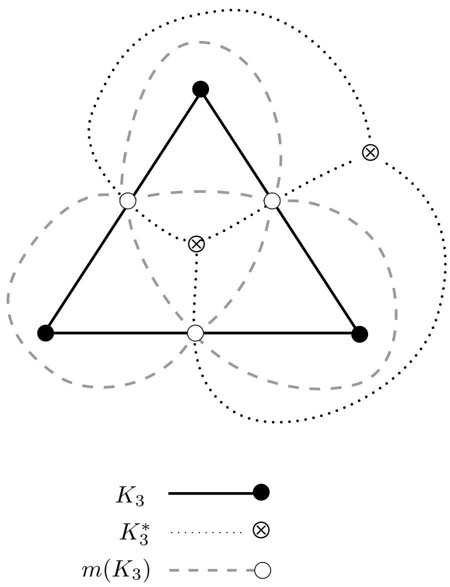


Figure 3: The medial graph (dashed grey lines) of K_3 (solid black lines) is isomorphic to the medial graph of the dual graph K_3^* (the graph K_3^* is shown with dotted black lines).

Question 3

(i) Can you find some more examples of self-dual plane graphs?

(ii) The *medial graph* $m(G)$ of a plane graph G is defined by placing vertices on the midpoints of edges of G and joining vertices by an edge when they lie on consecutive edges of a face (by a double edge if consecutive in two different faces). Prove that $m(G) \cong m(G^*)$. (See Figure 3.)

Theorem 2.1 *Let $G = (V, E)$ be a plane graph. Then the correspondence $e \mapsto e^*$ has the following properties:*

(i) $A \subseteq E$ is a cycle if and only if $A^* = \{e^* : e \in A\}$ is a cut in G^*.

(ii) $A \subseteq E$ is a cut if and only if $A^* = \{e^* : e \in A\}$ is a cycle in G^*.

To prove this rigorously requires appeal to the Jordan Curve Theorem.

Question 4

(i) Give an example of a 2-connected planar graph G with two inequivalent embeddings, i.e., find G with two embeddings that have non-isomorphic geometric duals G^*.

(ii) For any given k, describe a 2-connected planar graph with at least k pairwise inequivalent embeddings.

2.4 Matroids

Whitney [18] introduced matroids in 1935 as an abstraction of both linear independence and the properties of cycles in graphs. Matroids present a natural "self-dual" notion which captures both cycles and cutsets.

There are many "cryptomorphic" ways to define a matroid axiomatically. First we start with *independence* as the primitive notion – for graphs a set of edges is independent if it spans a forest, i.e., contains no cycles.

Definition 2.5 (Independent sets) *Let E be a finite set. A matroid M on E is a pair $(E, \mathcal{I})$ where $\mathcal{I}$ is a non-empty collection of subsets of E with the properties:*

(i) If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$ then $I_2 \in \mathcal{I}$ ($\mathcal{I}$ is an ideal),

(ii) (Exchange Property) *If $I_1, I_2 \in \mathcal{I}$, $|I_2| < |I_1|$, then there exists $e \in I_1$ such that $I_2 \cup \{e\} \in \mathcal{I}$.*

If E is a family of vectors in a vector space V, and $\mathcal{I}$ is the set of linearly independent subsets of E, then $(E, \mathcal{I})$ is a matroid (called a *vector matroid*).

A *basis* is a maximal independent set with respect to inclusion, i.e., a subset of edges that is independent with the property that adding any other edges destroys the property of independence.

The *rank* of $A \subseteq E$ is defined by

$$\rho(A) = \max\{|I| : I \in \mathcal{I}, \mathcal{I} \subseteq \mathcal{A}\}.$$

A *circuit* is a minimal non-independent set of edges with respect to inclusion, i.e., a subset of edges that is not independent but with the property that any proper subset is independent. Equivalently, a circuit is a minimal subset of edges contained in no basis. (A circuit in a graphic matroid corresponds to a spanning subgraph in which all vertices have degree 2 or 0.)

A matroid on E may be alternatively defined using any one of these three notions just defined as primitive:

Definition 2.6 (Circuits) *Let E be a finite set. A matroid M on E is a pair $(E, \mathcal{C})$ where $\mathcal{C}$ is a non-empty collection of subsets of E with the properties:*

(i) No member of $\mathcal{C}$ contains another ($\mathcal{C}$ is an antichain),

(ii) If $C_1, C_2 \in \mathcal{C}$ are distinct and $e \in C_1 \cap C_2$ there exists $C_3 \in \mathcal{C}$ such that $C_3 \subset C_1 \cup C_2$ and $e \notin C_3$.

Definition 2.7 (Rank function) *Let E be a finite set. A matroid M on E is a pair (E, ρ) where ρ is a function defined on subsets of E with the following properties:*

(i) $\rho(A) \leq |A|$ is a non-negative integer;

(ii) if $A \subseteq B \subseteq E$ then $\rho(A) \leq \rho(B)$ (ρ is monotone), and $\rho(\{x\}) \leq 1$;

(iii) (Semimodularity) *For any $A, B \subseteq E$,*

$$\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B).$$

Definition 2.8 (Bases) *Let E be a finite set. A matroid M on E is a pair $(E, \mathcal{B})$ where $\mathcal{B}$ is a non-empty collection of subsets of E with the properties:*

(i) No member of $\mathcal{B}$ contains another ($\mathcal{B}$ is an antichain);

(ii) (Steinitz-MacLane Exhange Lemma) *If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$ then there is $f \in B_2 \setminus B_1$ such that $B_1 \setminus \{e\} \cup \{f\} \in \mathcal{B}$.*

For more about matroids see for example Peter Cameron's notes at www.maths.qmul.ac.uk/˜pjc/comb/matroid.pdf (see in particular Section 3, in which the connection between the success of the greedy algorithm and bases of a matroid is explained), and for even more the books [15] and [12].

Question 5

(i) Prove that for any graph $G = (V, E)$ the collection $\mathcal{C}$ of all subsets of E that form a circuit in G forms a matroid on E.

What are the independent sets of this matroid? What is the corresponding rank function? And what are the bases?

Matroids which are defined in this way are called *graphic matroids*, usually denoted by $M(G)$. If $M(G)$ is defined by the cycles of G then we also speak about the *cycle matroid* of G.

(ii) Let E be a finite (multi)set of vectors (in a vector space). Subsets of E will be independent if they are linearly independent. Bases and rank function (defined as the dimension of the space generated by the set) have a clear meaning from linear algebra.

What are the circuits?

Matroids which are defined in this way are called *linear* or *representable matroids*. If the vector space is over field $\mathbb{F}$ then the matroid is $\mathbb{F}$*-representable*, and a *binary matroid* is one that is $\mathbb{Z}_2$-representable.

(iii) Prove that the lines of the Fano plane (see Figure 4) form the circuits of a matroid (called the *Fano matroid* and denoted by $Fano$). Alternatively, declare a set of points to be independent if it does not contain a line and prove that this defines a matroid (the same one of course). Is $Fano$ a binary matroid?

(iv) Let D be a digraph and S and T subsets of its vertices (not necessarily disjoint). Show that M defined on T is a matroid when it is stipulated that $I \subseteq T$ is independent if there exists a set of vertex-disjoint paths each starting in S and whose endpoints are exactly I. (This is a *gammoid*; a *strict gammoid* is the case where $S = T$ comprises all vertices of D.)

When are two matroids on sets E and E' isomorphic? One obvious thing to demand is a bijection $\iota : E \longrightarrow E'$ such that, for every $A \subseteq E$, (i) $\iota(A)$ is a circuit iff A is a circuit, (ii) $\iota(A)$ is independent iff A is independent, (iii) $\iota(A)$ is a basis iff A is a basis, and (iv) $\rho(A) = \rho(\iota(A))$.

But when do two graphs have isomorphic cycle matroids? This is more interesting and it leads to the following notion:

Definition 2.9 *Two graphs G and G' are* 2-isomorphic *if G can be transformed into G' by means of the following two operations and their inverses:*

(i) Identify two vertices in different connected components of G;

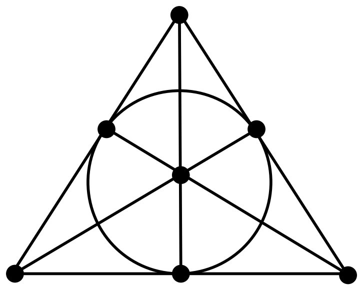


Figure 4: The Fano plane, consisting of 7 points and 7 lines, each containing 3 points. One line is represented as a circle.

(ii) Suppose G is obtained from disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2, and identifying v_1 of G_1 and v_2 of G_2. The Whitney twist *of G is the graph obtained by identifying u_1 with v_2 and u_2 with v_1.*

The first operation joins two components in a 1-cut (its inverse separating a graph at a 1-cut). The Whitney twist acts by flipping the graph G about one of its 2-cuts, and is illustrated in Figure 5.

Question 6 Suppose that the graphs G_1 and G_2 are connected planar graphs in Figure 5. Let G be the graph obtained by identifying u_1 with v_1 and u_2 with v_2, and G' the Whitney twist of G. Describe how $(G')^*$ is related to G^* in terms of the graphs G_1^* and G_2^*.

Proposition 2.10 *If two graphs are* 2*-isomorphic then their cycle matroids are isomorphic*

Proof. Clearly the edge set of cycles are unchanged when identifying two vertices in different components. Suppose G' is obtained from G by a Whitney twist about a given 2-cut of G. A cycle that does not pass through either vertex of the 2-cut remains unchanged. A cycle of G passing through one of the vertices of the 2-cut must pass through the other. If traversing this cycle we encounter the edges $e_1, e_2, \ldots, e_i, f_1, f_2, \ldots, f_j$, where the e-edges belong to G_1 and the f-edges to G_2, then in the Whitney twist corresponds the cycle in whose traversal we meet the edges in the order $e_1, \ldots, e_i, f_j, f_{j-1}, \ldots, f_1$. Thus the edge sets of cycles are the same in both graphs. □

Theorem 2.2 Whitney [17] *The cycle matroids of G and G' are isomorphic if and only if G and G' are 2-isomorphic. In particular, if G is 3-connected and G has isomorphic cycle matroid to G' then G and G' are isomorphic.*

Geometric duals of different embeddings of a plane graph G are 2-isomorphic, although they may not be isomorphic when G is not 3-connected.

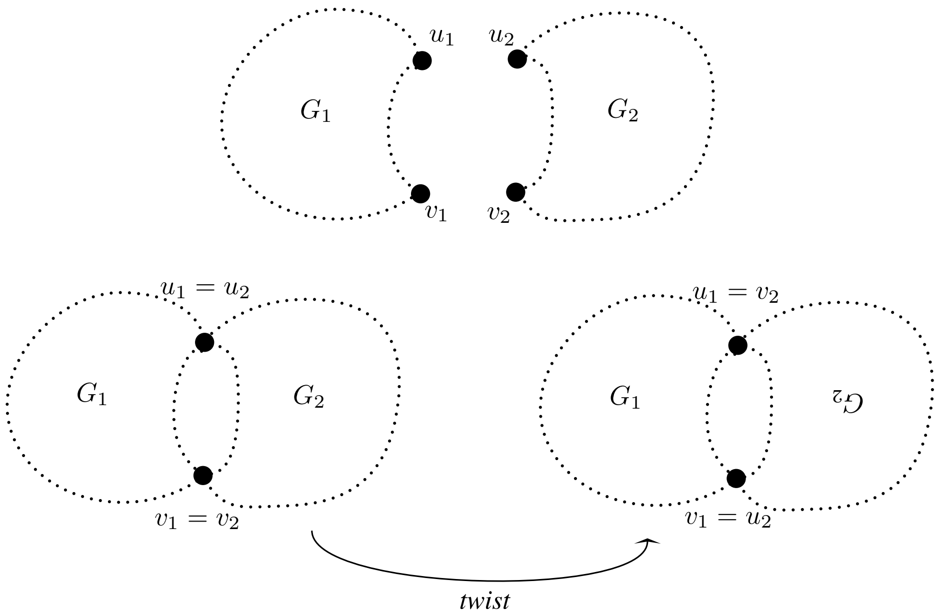


Figure 5: The Whitney twist of a graph about a 2-cut.

2.5 Dual matroids

Consider a graph $G = (V, E)$ and its cycle matroid $M(G)$. The independent sets of $M(G)$ correspond to the edge sets of spanning forests of G, the bases to maximal spanning forests of G. Thus the rank of the set E is $|V| - c(G)$ where $c(G)$ is the number of components of G and any basis of $M(G)$ has rank $|V| - c(G)$.

A maximal set of edges not containing any circuit (basis of G) is a maximal spanning forest of G. Circuits and cutsets are in dual correspondence for planar graphs. The dual notion of a basis for G is a maximal set of edges not containing any cutset of G, which is precisely the complement of a maximal spanning forest of G. Each of these sets has size $|E| - |V| + c(G)$.

Definition 2.11 *Let $M = (E, \mathcal{B})$ be matroid given by its bases. Then the* dual matroid *M^* is given by $(E, \mathcal{B}^*)$, where $\mathcal{B}^* = \{E \setminus B; B \in \mathcal{B}\}$.*

Question 7 Prove that M^* is indeed a matroid and that $M^{**} = M$.

Lemma 2.12 *Let M be a matroid on E with rank function ρ and M^* the dual of M, with rank function ρ^*. For $A \subseteq E$, let $A^* = E \setminus A$. Then,*

$$|A^*| - \rho^*(A^*) = \rho(E) - \rho(A),$$

and

$$\rho^*(E) - \rho^*(A^*) = |A| - \rho(A).$$

PROOF. Let I be a maximal independent subset of A in M, and $I \cup J$ a basis ($J \subseteq A^*$ by maximality). Set $K = A^* \setminus J = E \setminus (A \cup J) \subseteq E \setminus (I \cup J)$. The set K is an independent subset of A^* (since $E \setminus (I \cup J)$ is a basis of M^*). We then have

$$\rho^*(A^*) \geq |K| = |A^*| - |J| = |A^*| - \rho(E) + \rho(A),$$

with $\rho(E) - \rho(A)$ independent elements in J. Dually,

$$\rho(A) \geq |A| - \rho^*(E) + \rho^*(A^*).$$

But $|A| + |A^*| = |E| = \rho(E) + \rho^*(E)$, so the two inequalities are in fact equalities. $\square$

The rank function ρ^* of the dual matroid is thus given in terms of the rank function ρ for M by

$$\rho^*(A) = |A| - \rho(E) + \rho(E \setminus A).$$

The dual of a graphic matroid need not be a graphic matroid.

Question 8

(i) Let the rank function be given by $\rho(A) = \min(|E|, r)$ (r a fixed positive integer). Determine the bases and circuits of the corresponding matroid. (This matroid called *uniform* matroid and it is denoted by U_m^r, where $m = |E|$.) What is its dual?

(ii) A matching in a graph is a subset of pairwise disjoint edges. Do matchings form a matroid on a set of edges?

2.6 Deletion and contraction

A *loop* of a matroid M is an element e such that $\{e\}$ is not independent (i.e., $\rho(\{e\}) = 0$), equivalently e which lies in no independent set, or in no maximal independent set (basis).

Dually, a *coloop* is an element e contained in every basis of M. A coloop in a connected graph is an edge whose removal disconnects the graph. (Such an edge is commonly called a *bridge* or *isthmus*.)

Let M be a matroid on a set E given by its set of circuits $\mathcal{C}$. For e not a loop, denote by $\mathcal{C}'$ those sets in $\mathcal{C}$ not containing e and for e not a coloop by $\mathcal{C}''$ sets of the form $C \setminus \{e\}$ where the circuit C contains e. It is easy to see that both sets $\mathcal{C}', \mathcal{C}''$ satisfy the axioms for the circuits of a matroid.

This matroid defined by $\mathcal{C}'$ is the matroid obtained by *deletion* of e (or *restriction* to $E \setminus \{e\}$), denoted by $M \backslash e$. For $\mathcal{C}''$ the matroid is that obained by *contraction* of e, denoted by M/e.

If M is the cycle matroid of a graph $G = (V, E)$ then, for an edge $e \in E$, $M \backslash e$ and $M \backslash A$ are the cycle matroid of the graph $G \backslash e$. The matroid M/e is the cycle matroid of the graph G/e obtained by contraction of edge e.

It is intuitively clear (but involves the Jordan Curve Theorem) that if G is a plane graph then contraction of an edge e in G corresponds to deletion of edge e^* in the dual graph G^* and that the deletion of an edge e in G corresponds to contraction of edge e^* in G^*. This duality holds in general for matroids and their duals:

Proposition 2.13 *(i) e is a loop in M if and only if e is a coloop in M^*, and vice versa.*

(ii) If e is not a loop then $(M/e)^* \cong M^* \backslash e$.

(iii) If e is not a coloop then $(M \backslash e)^* \cong M^*/e$.

Proof. The element e lies in every basis of M (i.e., e is a coloop of M) if and only if it lies in no basis of M^* (i.e., e is a loop of M^*), and dually.

Suppose that e is not a loop of M. The bases of M/e are the bases of M containing e with e removed. The complement of such a basis in $E \setminus \{e\}$ is a basis of M^* not containing e, which is to say a basis of $M^* \backslash e$. So $(M/e)^* \cong M^* \setminus e$. Statement (iii) is proved dually. $\square$

Definition 2.14 *A matroid M' is a* minor *of matroid M if M' can be obtained from M by a sequence of contractions and deletions, which is denoted by $M' \prec M$.*

Using the above-mentioned facts we see that M' is a minor of M if and only if M'^* is a minor of M^*. Matroid minors feature in beautiful characterization theorems such as the following:

Theorem 2.3 *A matroid M can be represented by linear independence over the field $\mathbb{Z}_2$ (M is a* binary matroid*) if and only if U_4^2 fails to be a minor of M.*

Theorem 2.4 *A matroid M is the cycle matroid of a graph if and only if U_4^2, Fano and Fano* fail to be minors of M and the cycle matroids of $K_{3,3}$ and K_5 fail to be minors of the dual matroid M^*.*

Theorem 2.5 *A matroid M is representable by linear independence of vectors over every finite field (such matroids are called* regular*) if and only if U_4^2, Fano and Fano* fail to be minors of M.*

(The dual of the Fano matroid ought not be confused with its geometric dual when represented by a drawing in the plane.)

It follows that every graphic matroid (cycle matroid) is regular, but this can be seen much more easily directly. Another equivalent condition for regular matroids is that they can be represented by linear independence of columns of a totally unimodular matrix (a matrix whose square submatrices all have determinants equal to 0, 1, or -1). *Rota's conjecture* is that matroids representable over a fixed finite field are characterized by finitely many forbidden minors. In contrast with this, matroids in general and even matroids representable by independence of real vectors are not characterized by finitely many forbidden minors.

Planar graphs have a matroid characterization. In fact the connection between planar graphs (and thus to the Four Colour Conjecture) and linear algebra was a motivation for the concept of a matroid in the 1930s (H. Whitney, B. L. Van der Waerden).

Theorem 2.6 *A graph G is planar if and only if the dual matroid $M^*(G)$ of the cycle matroid $M(G)$ is graphic.*

Put otherwise, G is planar if and only if it has a dual graph. For duality of general graphs we need matroids. (Here the matroid *dual* is distinct from the *geometric dual*, defined for a 2-cell embedding of a graph in surfaces of arbitrary genus – genus 0 is the case of plane embedding, or, equivalently, embedding in the 2-dimensional sphere.)

3 The Tutte polynomial

3.1 Definitions

Let $G = (V, E)$ be a graph. A *spanning subgraph* is a subgraph (V, A) with $A \subseteq E$, and is denoted by G_A. An *induced subgraph* is a subgraph (U, A), where $A = \{uv \in E : u \in U, v \in U\}$, and is denoted by $G[U]$. The number of connected components of G is denoted by $c(G)$. A *maximal spanning forest* F is a forest which is a spanning subgraph of G with the property that F is contained in no other spanning forest of G, i.e., no edge of G can be added to F without creatng a cycle of G. A maximal spanning forest of a connected graph is a spanning tree.

The *rank* $r(G) = |V| - c(G)$ is the size of maximal spanning forest of G. Conversely, a spanning subgraph G_A with $c(G_A) = c(G)$ is a maximal spanning forest of G. For $A \subseteq E$ we often identify A with the spanning subgraph (V, A) and write $r(A)$ for $r(G_A)$. So $r(A) = |A|$ if and only if G_A is a spanning forest; $r(A) = r(E)$ if and only if G_A has the same number of connected components as G; and $r(A) = |A| = r(E)$ if and only if G_A is a maximal spanning forest of G. The *nullity* $n(G) = |E| - r(G)$ is the dimension of the cycle space of G (for a plane graph, this is the number of faces of G excepting the outer face). For $A \subseteq E$ we set $n(A) = n(G_A)$.

Deleting an edge $e \in E$ forms the spanning subgraph $G\backslash e = (V, E\backslash\{e\})$. *Contracting* an edge $e = uv$ forms the graph G/e obtained by deleting e and then identifying the endpoints of e.

An edge e is a *bridge* (*isthmus*, *cut-edge*, *coloop*) of G if $r(G\backslash e) < r(G)$, i.e., the number of connected components is increased upon removing e. An edge e is a bridge if and only if $r(\{e\}) = 1$. An edge $e = uv$ is a *loop* of G if $u = v$. Contracting a loop is the same as deleting it. An edge e is loop if and only if $n(\{e\}) = 1$. An edge e is *ordinary* if it is neither a bridge nor a loop.

Proposition 3.1 *Let $\tau(G)$ denote the number of maximal spanning forests of $G = (V, E)$. Then*

$$\tau(G) = \begin{cases} \tau(G/e) + \tau(G\backslash e) & e \text{ ordinary}, \\ \tau(G/e) & e \text{ a bridge}, \\ \tau(G\backslash e) & e \text{ a loop}, \\ 1 & G \text{ has no edges}. \end{cases} \quad (1)$$

Proof. Partition maximal spanning forests of G into two classes: those that contain e and those that don't. □

> **Question 9** Write down recurrences similar to (1) satisfied by
>
> (i) the number of spanning forests of G,
>
> (ii) the number of spanning subgraphs with the same number of connected components as G.

3.2 The chromatic polynomial

For $k \in \mathbb{N}$ the function $P(G; k)$ is defined to be equal to the number of proper k-colourings of G:

$$P(G; k) = \#\{f : V \to [k] : f(u) \neq f(v) \text{ for each } uv \in E\},$$

where $[k] = \{1, \ldots, k\}$.

Proposition 3.2 *For any edge $e \in E$,*

$$P(G; k) = \begin{cases} P(G\backslash e; k) - P(G/e; k) & e \text{ ordinary}, \\ (k-1)P(G/e; k) & e \text{ a bridge}, \\ 0 & e \text{ a loop}, \\ k^{|V|} & G \text{ has no edges}. \end{cases} \quad (2)$$

In fact, for any edge e –ordinary, bridge or loop– we have $P(G; k) = P(G\backslash e; k) - P(G/e; k)$.

> **Question 10** Use Proposition 3.2 to show that
>
> (i) $P(G; k)$ is a polynomial in k of degree $|V|$,
>
> (ii) the coefficient of k^i in $P(G; k)$ is non-zero for $c(G) \leq i \leq |V|$,
>
> (iii) the coefficients of k^i, $c(G) \leq i \leq |V|$, alternate in sign.

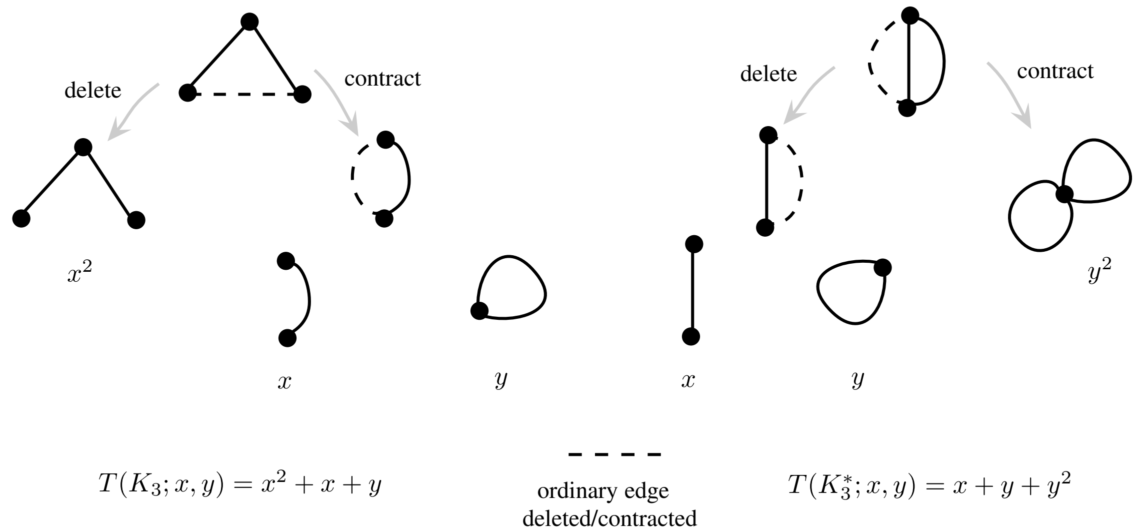


Figure 6: Using deletion-contraction to compute the Tutte polynomial of K_3 and its dual K_3^*.

3.3 Tutte polynomial by deletion-contraction

The Tutte polynomial of a graph $G = (V, E)$ can be defined recursively by

$$T(G; x, y) = \begin{cases} T(G/e; x, y) + T(G\backslash e; x, y) & e \text{ ordinary,} \\ xT(G/e; x, y) & e \text{ a bridge,} \\ yT(G\backslash e; x, y) & e \text{ a loop,} \\ 1 & G \text{ has no edges.} \end{cases} \tag{3}$$

Alternatively,

$$T(G; x, y) = \begin{cases} T(G/e; x, y) + T(G\backslash e; x, y) & e \text{ ordinary,} \\ x^k y^\ell & G \text{ consists of } k \text{ bridges and } \ell \text{ loops,} \end{cases} \tag{4}$$

A graph G is 2-*connected* if and only if has no cut-vertex. A loop on a single vertex (C_1) and a single bridge (K_2) are both 2-connected. For the case of many loops on a single vertex (where one might still consider the vertex not to be a cut-vertex) we refer to the cycle matroid, which is the direct sum of its constituent loops: so this graph is not 2-connected when there is more than one loop.

A *block* of G is a maximal 2-connected induced subgraph of G. If G is not 2-connected then it can be written in the form $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| \le 1$. The intersection graph of the blocks of a loopless connected graph is a tree. In particular, if G is loopless and connected and has at least two blocks then there are at least two *endblocks* of G which are blocks containing only one cut-vertex of G.

Proposition 3.3 *The Tutte polynomial of G is multiplicative over the connected components of G and over the blocks of G: if $G = G_1 \cup G_2$ where G_1 and G_2 share at most one vertex then $T(G_1 \cup G_2; x, y) = T(G_1; x, y)T(G_2; x, y)$.*

PROOF. The statement is true when each edge is either a bridge or a loop, since in this case $T(G; x, y) = x^k y^\ell$, where k is the number of bridges and ℓ the number of loops. We argue by induction on the number of ordinary edges of G. Let $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = 1$. The endpoints of any edge e must belong to the same block of G; if e is a bridge or loop then it forms its own block. Suppose $G = G_1 \cup G_2$ where G_1 is a block of G containing an ordinary edge e. Deleting or contracting e can only decrease the number of ordinary edges of G and since e is ordinary we have, writing $T(G; x, y) = T(G)$,

$$\begin{aligned} T(G) &= T(G/e) + T(G\backslash e) \\ &= T(G_1/e \cup G_2) + T(G_1\backslash e \cup G_2) \\ &= [T(G_1/e) + T(G_1\backslash e)]T(G_2) \\ &= T(G_1)T(G_2), \end{aligned}$$

where to obtain the third line we applied the inductive hypothesis. □

Theorem 3.4 "Recipe Theorem" *Let $\mathcal{G}$ be a minor-closed class of graphs. There is a unique graph invariant $f : \mathcal{G} \to \mathbb{Z}[x, y, \alpha, \beta, \gamma]$ such that for graph $G = (V, E)$*

$$f(G) = \begin{cases} \alpha f(G/e) + \beta f(G\backslash e) & e \text{ ordinary edge of } G, \\ x f(G/e) & e \text{ a bridge in } G, \\ y f(G\backslash e) & e \text{ a loop in } G, \\ \gamma^{|V|} & G \text{ has no edges.} \end{cases} \tag{5}$$

The graph invariant f is equal to the following specialization of the Tutte polynomial:

$$f(G) = \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \tag{6}$$

NOTE. (i) If instead of contracting a bridge we require that $f(G) = xf(G\backslash e)$ when e is a bridge, the Tutte polynomial is evaluated at the point $(\gamma x/\alpha, y/\beta)$ instead of $(x/\alpha, y/\beta)$. In particular, when $\gamma = 1$ it does not matter whether bridges are deleted or contracted.

(ii) If either α or β is zero then we interpret (6) as the result of substituting values of the parameters after expanding the expression on the right-hand side as a polynomial in $\mathbb{Z}[\alpha, \beta, \gamma, x, y]$. Given a graph G with k bridges and ℓ loops, if $\alpha = 0$ then $f(G) = \gamma^{c(G)} \beta^{n(G)-\ell} x^{r(G)} y^\ell$, and if $\beta = 0$ then $f(G) = \gamma^{c(G)} \alpha^{r(G)-k} x^k y^{n(G)}$. If both α and β are zero then $f(G) = 0$ if G has an ordinary edge, while $f(G) = \gamma^{c(G)} x^k y^\ell$ if $E(G)$ consists of just k bridges and ℓ loops.

PROOF. Uniqueness of $f(G)$ follows by induction on the number of edges and application of the recurrence (5).

Formula (6) is certainly true for cocliques $\overline{K}_n$. If G consists just of k bridges and ℓ loops and has c connected components, then $f(G) = \gamma^c x^k y^\ell$ and since $r(G) = k$ and $n(G) = \ell$ we have $T(G; \frac{x}{\alpha}, \frac{y}{\beta}) = \left(\frac{x}{\alpha}\right)^k \left(\frac{y}{\beta}\right)^\ell$, so (6) is satisfied. Let e be an ordinary edge, and note that $c(G) = c(G/e) = c(G\backslash e)$, so that $r(G/e) = r(G) - 1$, $r(G \setminus e) = r(G)$ and $n(G/e) = n(G)$, $n(G\backslash e) = n(G) - 1$. By induction on the number of ordinary edges,

$$\begin{aligned} f(G) &= \alpha f(G/e) + \beta f(G\backslash e) \\ &= \alpha \cdot \gamma^{c(G)} \alpha^{r(G)-1} \beta^{n(G)} T(G/e; \frac{x}{\alpha}, \frac{y}{\beta}) + \beta \cdot \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)-1} T(G\backslash e; \frac{x}{\alpha}, \frac{y}{\beta}) \\ &= \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \end{aligned}$$

□

Proposition 3.5 *The chromatic polynomial is given by*

$$P(G; z) = (-1)^{r(G)} z^{c(G)} T(G; 1 - z, 0).$$

More generally, the monochrome polynomial*,*

$$B(G; k, y) = \sum_{f : V(G) \to [k]} y^{\#\{uv \in E(G) : f(u) = f(v)\}},$$

is the following specialization of the Tutte polynomial:

$$B(G; k, y) = k^{c(G)} (y - 1)^{r(G)} T(G; \frac{y - 1 + k}{y - 1}, y).$$

Proof. For the chromatic polynomial we have $P(G; z) = (z - 1)P(G/e; z)$ when e is a bridge, for we have $P(G\backslash e; z) = zP(G/e; z)$. A direct argument for $P(G \setminus e; k) = kP(G/e; k)$ when $e = uv$ is a bridge is as follows. Suppose $G\backslash e = G_1 \cup G_2$ with $u \in V(G_1)$ and $v \in V(G_2)$. Then G/e is obtained from $G_1 \cup G_2$ by identifying the vertices u and v to make a cut-vertex w. Given a fixed colour $\ell \in [k]$, there are $P(G_1; k)/k$ proper colourings $f_1 : V(G_1) \to [k]$ of G_1 with $f_1(w) = \ell$, and $P(G_2; k)/k$ proper colourings $f_2 : V(G_2) \to [k]$ of G_2 with $f_2(w) = \ell$. Since there are no edges between G_1 and G_2, there are $P(G_1; k)P(G_2; k)/k^2$ proper colourings of G/e with $f(w) = \ell$. This number is independent of ℓ, so there are $P(G_1; k)P(G_2; k)/k$ proper colourings of G/e. On the other hand, there are $P(G_1; k)P(G_2; k)$ proper colourings of $G\backslash e$. Hence $P(G\backslash e; k) = kP(G/e; k)$ when e is a bridge of G.

A similar argument to the recurrence for the chromatic polynomial (Proposition 3.2) gives

$$B(G; k, y) = (y - 1)B(G/e; k, y) + B(G\backslash e; k, y), \tag{7}$$

valid for all edges e. When e is a bridge we have $B(G\backslash e; k, y) = kB(G/e; k, y)$, by a similar argument to the chromatic polynomial, by conditioning on the colour of the cut-vertex w

of G/e obtained by identifying the endpoints of e. Instead of proper colourings, consider colourings with exactly m_1 monochrome edges in G_1 and exactly m_2 monochrome edges in G_2. Then the number of such colourings for $G\backslash e$ (the disjoint union of G_1 and G_2) is k times the number for G/e (the gluing of G_1 and G_2 at a vertex). Collecting together all colourings for which $m_1 + m_2 = m$, this implies that the coefficient of y^m in $B(G\backslash e; k, y)$ is equal to k times the corresponding coefficient in $B(G/e; k, y)$. Since this holds for each m, it follows that $B(G\backslash e; k, y) = kB(G/e; k, y)$ when e is a bridge, and so $B(G; k, y) = (y-1+k)B(G/e)$ by the recurrence formula (7). When e is a loop $B(G; k, y) = yB(G\backslash e; k, y)$ since a loop is always monochromatic (or by looking at the recurrence formula (7) with $G/e \cong G\backslash e$ when e is a loop).

The result now follows by Theorem 3.4. □

Remark. The monochrome polynomial is the partition function for the k-state Potts model in disguise.

3.4 Sugraph expansion of the Tutte polynomial

First let's recap some notation. Let $G = (V, E)$ be a graph and $A \subseteq E$. Identify A with the spanning subgraph $G_A = (V, A)$. The rank of A is defined by $r_G(A) = |V(G)| - c(G_A)$ (this is the matroid rank function on the cycle matroid of G). The nullity of A is defined by $n_G(A) = |A| - r_G(A)$. Thus $r_G(E) = r(G)$ and $n_G(E) = n(G)$ in the notation already introduced for the rank and nullity of the graph G. When context makes it clear what graph G is, we drop the subscript and write $r(A)$ for $r_G(A)$ and $n(A)$ for $n_G(A)$.

It is easy to see that $0 \leq r(A) \leq |A|$ with $r(A) = 0$ if and only if A is empty or a set of loops, and $r(A) = |A|$ if and only if G_A is a forest (set of bridges). Also, $A \subseteq B$ implies $r(A) \leq r(B)$ and $r(A) = r(E)$ if and only if $c(G_A) = c(G)$.

Proposition 3.6 *The Tutte polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{n(A)}. \tag{8}$$

PROOF. Set

$$R(G; u, v) = \sum_{A \subseteq E} u^{r(E)-r(A)} v^{|A|-r(A)},$$

(the Whitney rank-nullity generating function for G). We wish to prove that $T(G; x, y) = R(G; x-1, y-1)$ and shall do this by verifying that $R(G; u, v)$ satisfies the recurrence: (i) $R(G; u, v) = 1$ if $E = \emptyset$, (ii) $R(G; u, v) = (u+1)R(G\backslash e; u, v)$ when e is a bridge, (iii) $R(G; u, v) = (v+1)R(G\backslash e; u, v)$ when e is a loop, and (iv) $R(G; u, v) = R(G/e; u, v) + R(G\backslash e; u, v)$ when e is ordinary.

When $E = \emptyset$ we have $R(G; u, v) = 1$.

If $e \notin A$ then

$$r_G(A) = r_{G\backslash e}(A). \tag{9}$$

If $e \in A$ then

$$r_{G\backslash e}(A\backslash e) = \begin{cases} r_G(A) - 1 & \text{if } e \text{ is a bridge,} \\ r_G(A) & \text{if } e \text{ is a loop,} \end{cases} \tag{10}$$

and

$$r_{G/e}(A\backslash e) = r_G(A) - 1 \quad \text{if } e \text{ is ordinary or a bridge.} \tag{11}$$

Suppose e is a bridge. Then by (9) and (10),

$$\begin{aligned} R(G;u,v) &= \sum_{A\subseteq E\backslash e} u^{r_G(E)-r_G(A)} v^{|A|-r_G(A)} + \sum_{e\in A\subseteq E} u^{r_G(E)-r(A)} v^{|A|-r_G(A)} \\ &= u \sum_{A\subseteq E\backslash e} u^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)} v^{|A|-r_{G\backslash e}(A)} \\ &\qquad + \sum_{B=A\backslash e} u^{r_{G\backslash e}(E\backslash e)+1-(r_{G\backslash e}(B)+1)} v^{|B|+1-(r_{G\backslash e}(B)+1)} \\ &= (u+1)R(G\backslash e;u,v). \end{aligned}$$

The case when e is a loop is similarly argued.

When e is ordinary, by (9) and (11),

$$\begin{aligned} R(G;u,v) &= \sum_{A\subseteq E\backslash e} u^{r_G(E)-r_G(A)} v^{|A|-r_G(A)} + \sum_{e\in A\subseteq E} u^{r_G(E)-r_G(A)} v^{|A|-r_G(A)} \\ &= \sum_{A\subseteq E\backslash e} u^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)} v^{|A|-r_{G\backslash e}(A)} \\ &\qquad + \sum_{B=A\backslash e} u^{r_{G/e}(E\backslash e)+1-(r_{G/e}(B)+1)} v^{|B|+1-(r_{G/e}(B)+1)} \\ &= R(G\backslash e;u,v) + R(G/e;u,v). \end{aligned}$$

□

It is common to *define* the Tutte polynomial by its subgraph expansion (8), having over the deletion–contraction formulation (3) the advantage of being unambiguously well-defined. On the other hand, it is not apparent from (8) that the coefficients of the Tutte polynomial are non-negative integers, and often it is easier to derive a combinatorial interpretation for an evaluation of the Tutte polynomial by using the deletion–contraction recurrence. Nonetheless, it is easy to read off some evaluations of the Tutte polynomial from its subgraph expansion.

Question 11 Let $G = (V, E)$ be a connected graph. Using the subgraph expansion for $T(G; x, y)$ show the following:

(i) $T(G; 1, 1) = \#\text{spanning trees}$,
$T(G; 2, 1) = \#\text{spanning forests}$,
$T(G; 1, 2) = \#\text{connected spanning subgraphs}$,
and $T(G; 2, 2) = 2^{|E|} = \#\text{spanning subgraphs}$.

(ii) If $(x-1)(y-1) = 1$ then $T(G; x, y) = (x-1)^{r(E)} y^{|E|}$.

(iii) The generating function for spanning forests of G by number of connected components is given by

$$xT(G; x+1, 1) = \sum_{\substack{A \subseteq E \\ n(A)=0}} x^{c(G_A)}.$$

(iv) The generating function for connected spanning subgraphs of G by size is given by

$$y^{|V|-1} T(G; 1, y+1) = \sum_{\substack{A \subseteq E \\ c(G_A)=c(G)}} y^{|A|}.$$

Along the hyperbola $(x-1)(y-1) = z$ we have, for graph $G = (V, E)$,

$$\begin{aligned} T(G; x, y) &= (y-1)^{-|V|} \sum_{A \subseteq E} \left(\frac{z}{y-1} \right)^{c(G_A)-c(G)} (y-1)^{|A|+c(G_A)} \\ &= (y-1)^{-r(G)} z^{-c(G)} \sum_{A \subseteq E} z^{c(G_A)} (y-1)^{|A|}. \end{aligned}$$

When $y = 0$ this is the subgraph expansion for the chromatic polynomial obtained by an inclusion–exclusion argument. The polynomial $\sum_{A \subseteq E} z^{c(G_A)} w^{|A|}$ is the partition function for the Fortuin–Kasteleyn random cluster model in statistical physics (the normalizing constant for a probability space on subgraphs of G, the probability of $G_A = (V, A)$ depending on both $|A|$ and $c(A)$). This model generalizes the k-state Potts model, which is the case $z = k \in \mathbb{Z}_+$, and whose partition function we have already met in the form of the monochrome polynomial $B(G; k, y)$.

3.5 Tutte polynomial of planar graphs

Let $G = (V, E, F)$ be a connected plane graph, with set of faces F, and let $G^* = (V^*, E^*, F^*)$ be its geometric dual. To construct G^*, put a vertex in the interior of each face of G, and connect two such vertices of G^* by edges that correspond to common boundary edges between the corresponding faces of G. If there are several common boundary edges the result is a multiple edge of G^*.

We identify V^* with F, E^* with E, and F^* with V.

Proposition 3.7 *If G is a connected planar graph with dual G^* then $T(G^*; x, y) = T(G; y, x)$.*

Proof. A bridge in G is a loop in G^*, a loop in G is a bridge in G^*, and deleting (contracting) an edge in G corresponds to contracting (deleting) an edge in G^*. In other words, $(G/e)^* \cong G^*\backslash e$ and $(G\backslash e)^* \cong G^*/e$. From these properties, that $T(G^*; x, y) = T(G; y, x)$ follows from the deletion-contraction recurrence for the Tutte polynomial. □

A subgraph of G on edges $A \subseteq E$ has no cycles (i.e., is a forest) if and only if the subgraph in the dual G^* on edges $E \setminus A$ is connected. If there is a cycle in A then its edges form the boundary of a simple closed curve in the plane, inside which lies at least one vertex of G^* (corresponding to a face enclosed by the cycle) and outside of which lies another vertex of G^*. Likewise, the edges of A form a connected subgraph of G if and only if the edges of $E \setminus A$ form a forest of G^*: any cycle in G^* has to cross an edge of a connected subgraph A.

Question 12

(i) Prove that the rank and nullity functions of a planar graph and its dual are related by $r_{G^*}(A) = n_G(E) - n_G(E \setminus A) = |A| - r_G(E) + r_G(E \setminus A)$, and $n_{G^*}(A) = r_G(E) - r_G(E \setminus A) = |A| - n_G(E) + n_G(E \setminus A)$.

(ii) Deduce that $T(G^*; x, y) = T(G; y, x)$ by using the subgraph expansion of the Tutte polyomial.

4 Medial graphs and 2-in 2-out digraphs

4.1 Medial of a plane graph

To form the medial graph $m(G)$ of a connected plane graph G that has at least one edge first place a vertex v_e into the interior of each edge e of G. Then, for each face F of G, join v_e and v_f by an edge lying in F if and only if the edges e and f are consecutive on the boundary of F. The medial graph $m(G)$ is 4-regular, as each face creates two adjacencies for each edge on its boundary. The faces of $m(G)$ divide naturally into two types: those that contain vertices of G (vertex-faces), and those corresponding to faces of G (face-faces). Vertex-faces will be coloured black and face-faces coloured white. See Figure 7.

If G^* is the planar dual of G then $m(G^*) \cong m(G)$ (if $e \mapsto e^*$ is the duality mapping between edges of G and edges of G^* then e and f are consecutive edges of a face in G if and only if e^* and f^* are consecutive edges in a face of G^*).

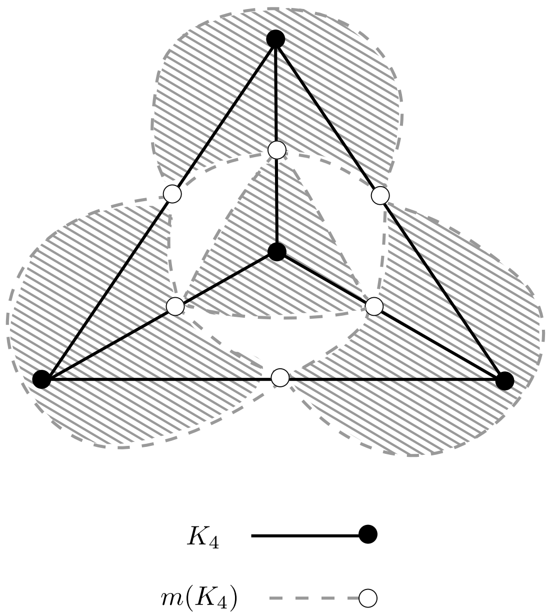


Figure 7: K_4 and its medial graph, with faces containing vertices of G shaded black.

The plane graph G is the *black face graph* of $m(G)$, i.e., the graph whose vertices are the black faces of $m(G)$ and whose edges join two black faces of $m(G)$ that share a vertex. The plane graph G^* is the white face graph of $m(G)$. The embedding of $m(G^*)$ differs from that of $m(G)$ in having a different outer face: black faces in one (vertices of G) become white faces in the other (faces of G).

Forming the black face graph is inverse to the medial construction. A 4-regular connected plane graph H has bipartite dual graph so we can always 2-colour the faces of H properly with colours black and white, making the exterior face white. If $G(H)$ is the black face graph of H then $m(G(H)) = G$.

Question 13 Describe how to construct the graph $m(G)^*$. What type of graph is it?

4.2 Eulerian tours of digraphs

Definition 4.1 *Let D be a digraph. An* Euler cycle *of D is a closed trail in D, i.e., a closed walk in which each edge of D is traversed at most once. An* Euler tour[4] *of D is a closed trail that traverses all the edges of D, i.e., an Euler cycle in which each edge of D*

[4]In [2] the term *Euler cycle* is used, for both Euler tour and Euler cycle. In [3] the term *Euler circuit* is used both for *Euler tour* and *Euler cycle*, the distinction not being made between tour of the graph and tour of a subgraph. The use of the word *cycle* for digraphs has been chosen here to correspond with that of

is traversed exactly once. In other words, a walk $v_1, e_1, v_2, e_2, \ldots, v_m e_m, v_1$ where $e_1, \ldots, e_m$ comprises a list of all the edges of D with no repetitions, and e_i is the edge directed from v_i to v_{i+1} (in which $v_{m+1} = v_1$).

Definition 4.2 *An* Euler cycle k-partition *of D is a partition of the edges of D into k non-empty Euler cycles $C_1, \ldots, C_k$. The number of Euler cycle k-partitions of D is denoted by $e_k(D)$. (Thus $e_1(D)$ is the number of Euler tours of D.)*

For an Euler tour C, let $\mathsf{ve}(C) = \begin{pmatrix} v_1 & e_1 & v_2 & e_2 & \ldots & v_m & e_m \end{pmatrix}$ denote the cyclic word comprising vertices and edges visited when traversing C (tours and cycles are considered equivalent up to starting point). The word $\mathsf{ve}(C)$ has the property that vertex v appears exactly $d^+(v)$ times and each edge appears exactly once.

Similarly, we denote by $\mathsf{v}(C) = \begin{pmatrix} v_1 & v_2 & \ldots & v_m \end{pmatrix}$ the cyclic word comprising vertices in the order visited when traversing C. The word $\mathsf{v}(C)$ has the property that vertex v appears exactly $d^+(v)$ times and each edge appears exactly once.

If there are no parallel directed edges in D then $\mathsf{v}(C)$ determines $\mathsf{ve}(C)$ uniquely.

Given an Euler tour C, the digraph D is uniquely determined by C. The vertex-edge word $\mathsf{ve}(C)$ contains not only the vertex-edge incidences and edge directions that determine D, but also the fact that this is an Euler tour, so C in fact contains more information than the adjacency matrix for D. For given C, we thus write $D = D(C)$ to emphasize that the digraph D is determined by C.

A digraph D has an Euler tour if and only if each vertex has the same indegree as outdegree, $d^+(v) = d^-(v)$, one of the orginating results of graph theory, due to Euler [5]. The vertex v is encountered exactly $d^+(v)$ times in traversing an Euler tour of D.

The algorithms of Fleury and Hierholzer for constructing Euler tours of a graph G, described in Section 2.1 above, suggest two ways to think of an Euler tour of a digraph D.

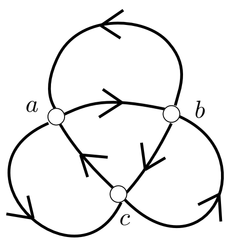
The first (corresponding to Hierholzer's algorithm) is to construct C by gluing together Euler cycles $C_1, \ldots, C_k$ of D, where C_1 is arbitrary, and C_{i+1} is a cycle that uses a vertex appearing in at least one of the cycles $C_1, \ldots, C_i$. In other words, the cycles of an Euler k-partition of D can be glued together to form an Euler tour of D. Given two cyclic vertex words $\begin{pmatrix} v & x \end{pmatrix}$ and $\begin{pmatrix} v & y \end{pmatrix}$, where x and y are sequences of vertices, representing Euler cycles C and C', the cyclic vertex word $\begin{pmatrix} v & x & v & y \end{pmatrix}$ represents the cycle obtained by gluing C and C' together, i.e., first traversing C staring at v and then, upon returning to the vertex v, following the cycle C'. By iterating this gluing procedure, all k cycles in an Euler k-partition of D glue together to form an Euler tour of D.

The second (corresponding to Fleury's algorithm) arises by fixing a starting point u for a given Euler tour and marking the last out-edge traversed from vertex $v \neq u$ before returning to u for the last time. These edges together form a spanning tree of D in which every vertex $v \neq u$ is connected to u by a directed path, in other words, a *spanning arboresecence* of D rooted at u. Conversely, given a spanning arboresence T of D rooted at u, an Euler tour can be traversed by first freely choosing one of $d^+(u)$ out-edges of u, and

its use for graphs (in which a circuit is minimal dependent set of edges, i.e., a closed path): the underlying graph of an Euler cycle of D is a cycle (Eulerian subgraph) of the underlying undirected graph of D.

Euler cycle partition

D



alternating orientation

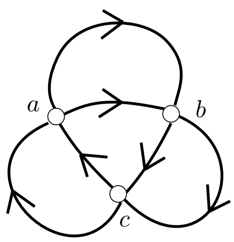
$e_1(D) = 3$ $\quad$ $(\, a \;\; b \;\; c \;\; a \;\; c \;\; b \,)$

$e_2(D) = 4$ $\quad$ $(\, a \;\; b \;\; c \;\; b \,)(\, a \;\; c \,)$

$(\, a \;\; b \;\; c \,)(\, a \;\; c \;\; b \,)$

$e_3(D) = 1$ $\quad$ $(\, a \;\; b \,)(\, b \;\; c \,)(\, c \;\; a \,)$

(Permutations of a, b, c produce the given number of Euler cycle partitions.)



anti-alternating orientation

$e_1(D) = 4$ $\quad$ $(\, a \;\; b \;\; c \;\; a \;\; b \;\; c \,)$

$e_2(D) = 4$ $\quad$ $(\, a \;\; b \;\; c \,)(\, a \;\; b \;\; c \,)$

(Choice between parallel arcs produces 4 Euler cycle partitions each.)

Figure 8: Two orientations of $m(K_3)$ and their Euler cycle partitions, in which the Euler cycles are presented as cyclic vertex words. An Eulerian orientation of 4-regular graph is *alternating* if at each vertex incoming edges alternate with outgoing edges, and *anti-alternating* of at each vertex incoming edges do not alternate with outgoing edges.

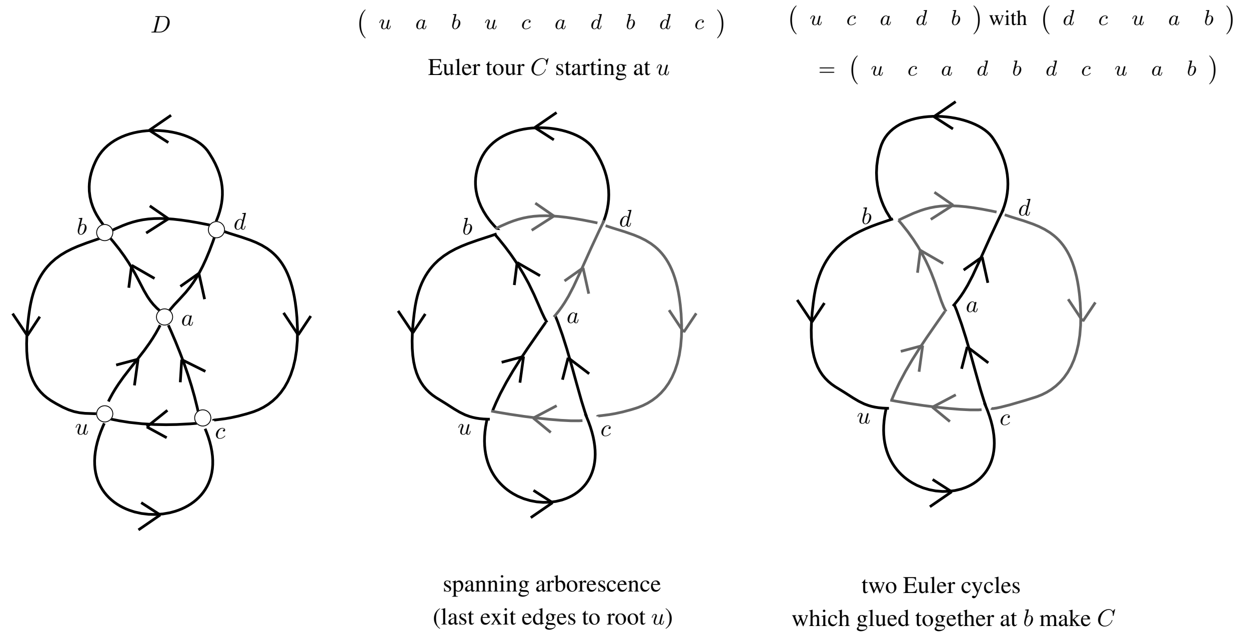


Figure 9: Euler tour C of 2-in 2-out digraph D starting from u corresponds to spanning arborescence rooted at u, whose edges are the second out-edge taken from $v \neq u$ when traversing C. The tour C may also be constructed by gluing together Euler cycles (illustrated here is one possibility, gluing the two given cycles at vertex b, which produces a shifted version of C).

then at each vertex v freely choosing any out-edge not on T, as long as there are any, and only when the edge on T remains taking it. This gives, for each spanning arboresecence T rooted at u,

$$d^+(u)!\prod_{v\neq u}(d^+(v)-1)!$$

Euler tours starting at a given outedge from u, i.e.,

$$\prod_{v}(d^+(v)-1)!$$

Euler tours. Moreover, an Euler tour corresponding to spanning arborescence T rooted at u cannot equal an Euler tour corresponding to a different spanning arborescence T' rooted at u.

By the Matrix Tree Theorem for digraphs, the number of spanning arborescences of D rooted at u is given by

$$t_u(D) = \det L_{V\setminus\{u\}},$$

where L is the Laplacian matrix $L = \Delta - A$, in which A is the adjacency matrix of D, with (v, w) entry equal to the number of directed edges from v to w, and Δ is the diagonal matrix with (v,v) entry equal to $d^+(v)$, and $L_{V\setminus\{u\}}$ is the matrix L with row and column indexed by u removed. (Alternatively, $t_u(D) = \frac{1}{n}\lambda_1\cdots\lambda_{n-1}$, where $\lambda_1,\ldots,\lambda_{n-1}$ are the non-zero eigenvalues of L.) It is perhaps surprising that the number of spanning arborescences of D does not depend on the root u.[5]

Theorem 4.3 ("BEST Theorem", [1, 13]) *The number of Euler tours of digraph $D = (V,E)$ is given by*

$$e_1(D) = t_u(D)\prod_{v\in V}(d^+(v)-1)!,$$

where u is an arbitrary vertex of D and $t_u(D)$ the number of spanning arborescences of D rooted at u.

4.3 2-in 2-out digraphs

From now on D will be a *2-in 2-out digraph*, i.e., each vertex has indegree 2 and outdegree 2. If D has n vertices then it has $2n$ edges.

For an Euler tour C of 2-in 2-out digraph D, the word $\mathsf{ve}(C)$ has the property that each vertex occurs exactly twice and each edge exactly once. For an Euler cycle the corresponding cyclic vertex-edge word has the property that each vertex occurs at most twice and each edge at most once. Similarly, the cyclic vertex word $\mathsf{v}(C)$ has the property that each vertex occurs exactly twice and for an Euler cycle the corresponding cyclic vertex word has the property that each vertex occurs at most twice.

For a 2-in 2-out digraph D there is a one-one correspondence between Euler tours of D and spanning arborescences of D rooted at a fixed vertex u. See Figure 9. However, we shall concentrate on the viewpoint of Euler cycle k-partitions (whose cycles glued together form an Euler tour).

Given an an undirected graph $G = (V,E)$, an Eulerian orientation of G is an orientation of the edges with the property that each vertex has as many incoming edges as outgoing edges: $d^-(v) = d^+(v)$ for each $v \in V$. When orienting edges according to a traversal of an Euler tour of G the result is an Eulerian orientation. Counting the number of Eulerian orientations of G is $\#\mathcal{P}$-complete [11], even for plane 4-regular graphs [7]. (The situation is quite different for a given digraph D, where by Theorem 4.3 counting Eulerian tours of D can be done in polynomial time.)

Proposition 4.4 *If G is a connected 4-regular graph on n vertices then G has $(-1)^{n-1}T(G;0,-2)$ Eulerian orientations.*

Note that this does not extend to graphs G generally: the number of Eulerian orientations of G is only given by this evaluation of the Tutte polynomial when all its vertex degrees belong to $\{0,1,2,4\}$.

[5]A similar phenomenon occurs for acyclic orientations of a connected graph G with unique sink at vertex u: this turns out to be independent of u and is in fact given by the Tutte polynomial evaluation $T(G;1,0)$.

PROOF. The easiest way to prove this relies on knowing what a nowhere-zero $\mathbb{Z}_3$-flow is and establishing that $(-1)^{n-1}T(G;0,-2) = F(G;3)$ is the number of nowhere-zero $\mathbb{Z}_3$-flows of connected 4-regular G. (Making an inductive deletion-contraction argument and applying the Recipe Theorem for evaluations of the Tutte polynomial is thwarted by the fact that the property of being 4-regular is destroyed by edge deletion and contraction.) $\square$

Definition 4.5 *Let C be an Euler cycle of D with cyclic vertex-edge word*

$$\mathsf{ve}(C) = \begin{pmatrix} x & a & y & b & x' & a & y' & b \end{pmatrix},$$

in which the vertices a, b are interlaced, and x, y, x', y' are vertex-edge sequences (possibly empty). The transposition *of C along a and b is the Euler cycle C^{ab} of D defined by the cyclic vertex-edge word*

$$\mathsf{ve}(C^{ab}) = \begin{pmatrix} x & a & y' & b & x' & a & y & b \end{pmatrix}.$$

Note that transposition along an interlaced pair in an Euler cycle C produced another Euler cycle of the same size; in particular, an Euler tour upon transposition becomes another Euler tour.

If the cyclic vertex word for C is $\mathsf{v}(C) = \begin{pmatrix} x & a & y & b & x' & a & y' & b \end{pmatrix}$, then that for C^{ab} is $\mathsf{v}(C^{ab}) = \begin{pmatrix} x & a & y' & b & x' & a & y & b \end{pmatrix}$. If D has no parallel edges then transposition can be defined in terms of vertex words rather than vertex-edge words, because in this case each vertex word $\mathsf{v}(C)$ uniquely determines the vertex-edge word $\mathsf{ve}(C)$. For the sake of simplicity we shall work with vertex words rather than vertex-edge words, with the understanding that only minor modifications need to be made in order to incorporate the case of parallel directed edges.

Definition 4.6 *Let a be a vertex of a 2-in 2-out digraph D such that (u, a), (a, v), (u', a), (a, v') are the directed edges of D incident with a. (Possibly $u = v$ or $u' = v'$, corresponding to loops on a.) A* transition *at a is one of the two possible pairings $\{u, v\}, \{u', v'\}$ or $\{u, v'\}, \{u', v\}$. (For the first pairing, the pair of edges $(u, a), (a, v)$ and the pair of edges $(u', a), (a, v')$ both form directed paths, and similarly for the second pairing.)*

Remark then that a 2-in 2-out digraph D on n vertices has 2^n Euler cycle partitions, corresponding to the independent choice of two possible transitions at each vertex of D.

Let C be an Euler cycle. When $\mathsf{v}(C) = \begin{pmatrix} \dots & u & a & v & \dots & u' & a & v' & \dots \end{pmatrix}$ the transition of C at a is $\{u, v\}, \{u', v'\}$.

Given $\mathsf{v}(C) = \begin{pmatrix} \dots & u & a & v & \dots & b & \dots & u' & a & v' & \dots & b \end{pmatrix}$, the transposition along interlaced a and b is given by

$$\mathsf{v}(C^{ab}) = \begin{pmatrix} \dots & u & a & v' & \dots & b & \dots & u' & a & v & \dots & b \end{pmatrix},$$

in which the transition at a has been switched from $\{u, v\}, \{u', v'\}$ to $\{u, v'\}, \{u', v\}$.

See Figure 10.

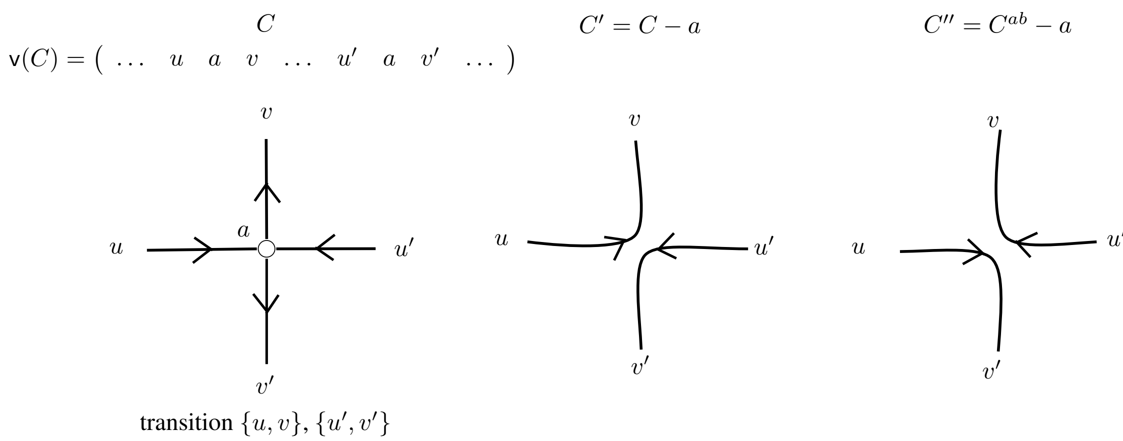


Figure 10: Transition at a vertex a, and contracting out a from an Euler tour.

Definition 4.7 *Suppose C is an Euler cycle with $\mathsf{v}(C) = \big(\ \dots \ u \ a \ v \ \dots \ u' \ a \ v' \ \dots \big)$. The* contraction *of C by a is the Euler cycle of $D\backslash\{a\}$ denoted by $C - a$ and given by*

$$\mathsf{v}(C-a) = \big(\ \dots \ u \ v \ \dots \ u' \ v' \ \dots \big).$$

Lemma 4.8 *If D has an Euler tour C with no interlaced pairs then D has only one Euler tour.*

Proof. Induction on the number of vertices of D. The base case is where D is a single vertex with two loops, for which the assertion is true. Suppose it is true for all 2-in 2-out digraphs on $n-1$ vertices and consider D on n vertices. By hypothesis D has an Euler tour C with no interlacements. This implies there is some vertex a with a loop. (Consider the word $\mathsf{v}(C)$ in which the two occurrences of a vertex u either enclose or are enclosed by a pair of occurrences of another vertex v, or the two occurrences are disjoint; for given vertex u, consider all the vertices that it encloses: either there are none, in which case take $a = u$, or there is a such a vertex v whose two occurrences are between those of u, and now repeat the argument with v in place of u and eventually an adjacent pair of vertices must be found.) The Euler tour $C - a$ of $D - \{a\}$ has no interlacements, hence $D - \{a\}$ has a unique Euler tour by induction hypothesis, and therefore so does D, as the only choice for an Euler tour at a is to traverse the loop between entering and leaving a. $\Box$

Lemma 4.9 *If C and C' are Euler tours of D then there is a sequence of transpositions that transforms C to C'. In other words, the orbit of an Euler tour of D under the action of transposition along interlaced vertices is the set of all Euler tours of D.*

Proof. The proof is by induction on the number of vertices of D. The case of a single vertex is vacuously true. Suppose the statement is true for digraphs on $n-1$ vertices and let D be a 2-in 2-out digraph on n vertices. Suppose that D has distinct Euler tours C and C'. If there is a vertex a at which C and C' have the same transition, then by contracting at a the tours $C - a$ and $C' - a$ can be obtained one from the other by a sequence of transpositions, which upon reinserting a means the same is true of C and C'.

Suppose then that C and C' have different transitions at all vertices. Then, since C and C' are distinct, by Lemma 4.8 there is an interlaced pair a and b in C. The Euler tour C^{ab} then has the same transition as C' at vertex a, and the previous argument shows that C^{ab} can be obtained from C' by a sequence of transpositions, and hence the same is true of $C = (C^{ab})^{ab}$. $\square$

5 Interlace polynomial

Definition 5.1 *The* interlace graph $H(C)$ *of an Euler tour of 2-in 2-out digraph D is defined on the vertex set of D traversed by C in which vertices a and b are adjacent if a and b are interlaced in C, i.e., if the cyclic vertex word of C takes the form*

$$\mathsf{v}(C) = \big(\ \ldots \quad a \quad \ldots \quad b \ldots \quad a \ldots \quad b \quad \ldots\ \big).$$

The interlace graph of C is the intersection graph of the *chord diagram* of C, in which the vertices of the cyclic vertex word of C are placed around a circle and each pair of like vertices is joined by a chord. This type of intersection graph is known as a *circle graph.* See Figure 11.

> **Question 14** Explain why the 5-wheel (the graph on six vertices formed by joining each vertex of a 5-cycle to a central vertex) is not a circle graph. (This is the smallest example of a graph that is not a circle graph.)

For a vertex a in graph H denote by $N(a)$ its open neighbourhood $\{c \in V(H) : ac \in E(H)\}$ and by $N[a]$ its closed neighbourhood $N(a) \cup \{a\}$.

Lemma 5.2 *Let a, b be interlaced in an Euler tour C of 2-in 2-out digraph D and C^{ab} the Euler tour of D obtained by transposition along a and b. Then the interlace graph $H(C^{ab})$ is obtained by applying the following two operations to $H(C)$:*

(i) Switch along ab: *toggle adjacencies between $N(a) \cap N(b)$, $N(a) \backslash N[b]$ and $N(b) \backslash N[a]$ (but not within these sets).*

(ii) Swap a and b, *i.e., ac is an edge in $H(C^{ab})$ if and only if bc is an edge in $H(C)$, and bc is an edge in $H(C^{ab})$ if and only if ac is an edge in $H(C)$).*

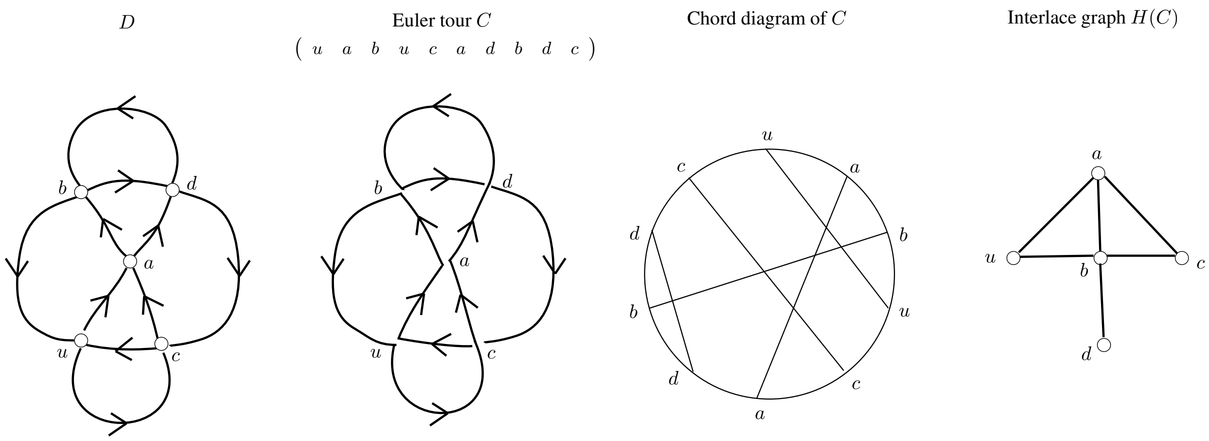


Figure 11: A 2-in 2-out digraph D with Euler tour C, its chord diagram, and the interlace graph of C, equal to the intersection graph of the chord diagram.

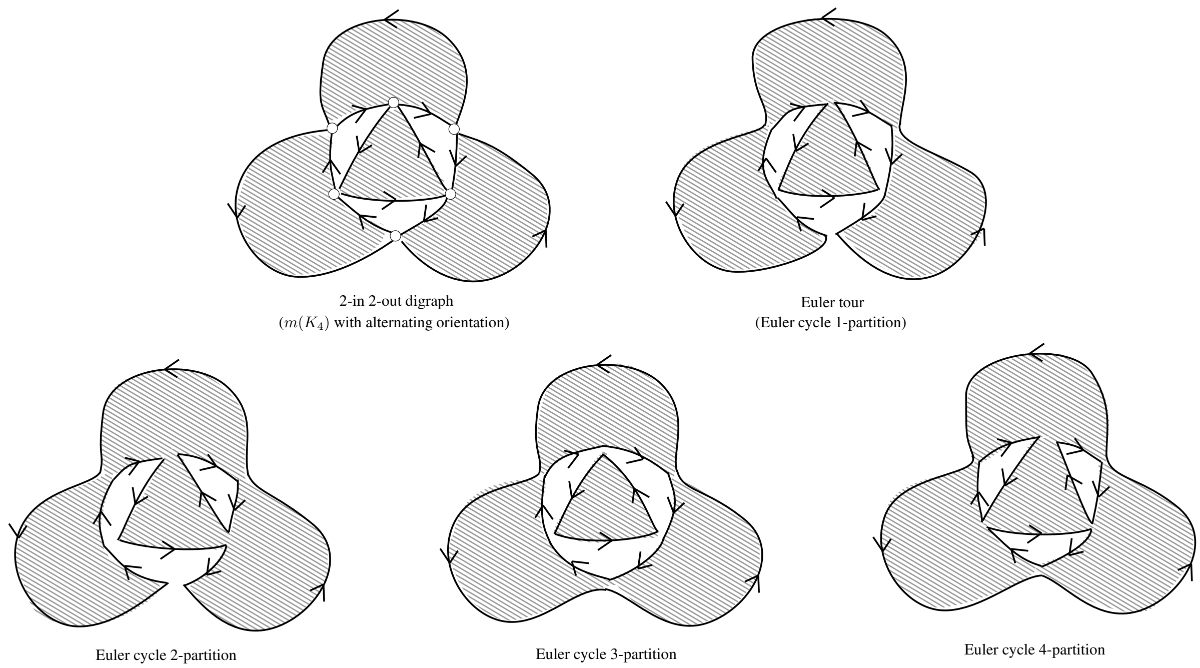


Figure 12: Canonical alternating orientation of $m(K_4)$ and examples of Euler cycle k-partitions for $k = 1, 2, 3, 4$.

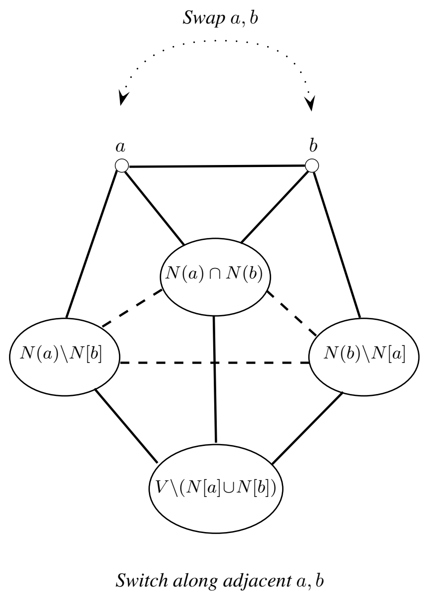


Figure 13: Switching along ab and then swapping a and b transforms $H(C)$ to $H(C^{ab})$. The dashed lines indicate where adjacencies need to be toggled. Remaining adjacencies in $H(C)$ are preserved.

Proof. [Case analysis, as indicated in lectures. See [3]. Details will be written up later.] □

See Figure 13.

Denote the graph obtained from H by switching along (adjacent) vertices a and b by H^{ab}, and the graph obtained from H by swapping a and b by H_{ab}. Then Lemma 5.2 says that for interlaced vertices a and b we have $H(C^{ab}) = (H(C)^{ab})_{ab}$. Since we shall only be concerned with the interlace graph $H(C)$ up to isomorphism, it suffices to work with the fact that $H(C^{ab}) \cong H(C)^{ab}$. Note also that $(H^{ab})^{ab} = H$. We say two simple graphs H and H' are *switching equivalent* if there is a sequence of switches transforming one into the other. By Lemma 5.2 the interlace graphs of Euler tours C and C' of a 2-in 2-out digraph D are switching equivalent.

Question 15

(i) Which 2-in 2-out digraphs have an Euler tour with empty interlace graph $\overline{K}_n$?

(ii) Show that K_n is unaffected by switching along an edge. Which digraphs D have an Euler tour with interlace graph K_n?

Define the function q_k on interlace graphs by $q_k(H(C)) = e_k(D(C))$, where C is an Euler tour of D. We have $q_1(K_1) = 1 = q_2(K_1)$ and $q_k(K_1) = 0$ for $k \geq 2$ since $D(C)$ in this case is the digraph on a single vertex with two loops.

Lemma 5.3 *The function q_1 satisfies the recurrence*

$$q_1(H(C)) = q_1(H(C) \setminus a) + q_1(H(C)^{ab} \setminus b),$$

when ab is an edge of $H(C)$, and $q_1(H(C)) = 1$ when $H(C)$ has no edges (C is the unique Euler tour of D).

PROOF. If $H(C)$ has no edges then by Lemma 4.8 there is a unique Euler tour and $q_1(H(C)) = 1 = e_1(D(C))$.

Otherwise, suppose ab is an edge of $H(C)$. Referring to Figure 10, let $D = D(C)$ be the 2-in 2-out digraph determined by C, $D' = D(C - a)$ that determined by the contraction of C at a and $D'' = D(C^{ab} - a)$ that determined by the contraction of C^{ab} at a. Since transposition at ab switches the transition at a, and $C \mapsto C^{ab}$ is a bijection on Euler tours of D, partitioning tours according to their transition at a we have

$$e_1(D(C)) = e_1(D(C - a)) + e_1(D(C^{ab} - a)).$$

The interlace graph of $D(C-a)$ is $H(C)\setminus a$ and the interlace graph of $D(C^{ab}-a)$ is $H(C)^{ab}\setminus b$ (where b is the vertex deleted since $H(C^{ab}) = (H(C)^{ab})_{ab}$ involves a swap of a and b which is not carried out in just the switch $H(C)^{ab}$). We have then $e_1(D(C - a)) = q_1(H(C) \setminus a)$ and $e_1(D(C^{ab} - a)) = q_1(H(C)^{ab} \setminus b)$ and the statement of the lemma is now proved. $\square$

Question 16 Prove that the function q_k, $k \geq 1$, satisfies the recurrence

$$q_k(H(C)) = q_k(H(C) \setminus a) + q_k(H(C)^{ab} \setminus b),$$

where C is an Euler tour of 2-in 2-out digraph D and a, b are interlaced in C.

Lemma 5.3, and the fact that the switching operation is defined on any simple graph, not just interlace graphs, prompted Arratia, Bollobás and Sorkin [3] to postulate the existence of a polynomial $Q(H; x)$ defined on simple graphs H as follows:

Definition 5.4 *The* interlace polynomial $Q(H; x)$ *of a simple graph $H = (V, E)$ is defined by the recurrence*

$$Q(H; x) = \begin{cases} Q(H \setminus a; x) + Q(H^{ab} \setminus b; x) & ab \in E \\ x^{|V|} & E = \emptyset. \end{cases}$$

Example 5.5 *Take $H = K_n$ and edge ab, for which $K_n \setminus a \cong K_n^{ab} \setminus b \cong K_{n-1}$. By the defining recurrence for $Q(K_n; x)$ we have, by induction on n, $Q(K_n; x) = 2^{n-1}x$.*

Note that $Q(H; 1) = q_1(H)$ when $H = H(C)$ is the interlace graph of an Euler tour C. The authors of [3] prove that the order of edges ab chosen in the switching and vertex-deletion recurrence defining $Q(H; x)$ does not affect the resulting polynomial, i.e., that the polynomial $Q(H; x)$ is well-defined. This is analogous to the situation for the recurrence defining the Tutte polynomial, where we have independence of the order of edge deletions and contractions. Also analogous to the case of the Tutte polynomial, it is possible to circumvent this somewhat tedious verification by producing a bona fide polynomial that does indeed satisfy the given recurrence.

Theorem 5.6 *The interlace polynomial of a simple graph $H = (V, E)$ is given by the induced subgraph expansion*

$$Q(H; x) = \sum_{U \subseteq V} (x-1)^{|U| - \mathrm{rk}(A_U)},$$

where A is the adjacency matrix of H, A_U its restriction to rows and columns indexed by U, and $\mathrm{rk}(A_U)$ the rank of the matrix A_U over $\mathbb{F}_2$ (where $\mathrm{rk}(A_\emptyset) = 0$ by fiat).

Proof. See [2, Ch. 9]. □

Proposition 5.7 *Switching equivalent graphs have the same interlace polyomial.*

Proof. For edge ab of H,

$$\begin{aligned} Q(H^{ab}; x) &= Q(H^{ba}; x) \\ &= Q(H^{ba} \setminus b; x) + Q(H \setminus a; x) \qquad \text{since } (H^{ba})^{ab} = H, \\ &= Q(H; x) \end{aligned}$$

□

Proposition 5.8 *The interlace polyomial is multiplicative over disjoint unions.*

Proof. We wish to prove that if $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are vertex-disjoint graphs then

$$Q(H_1 \cup H_2; x) = Q(H_1; x)Q(H_2; x).$$

If neither H_1 nor H_2 has any edges then the assertion is trivial, with $Q(H_1 \cup H_2; x) = x^{|V_1|+|V_2|} = x^{|V_1|}x^{|V_2|} = Q(H_1; x)Q(H_2; x)$. Without loss of generality then, suppose $ab \in E_1$. Then, by induction on the number of vertices of $H_1 \cup H_2$, and using the fact that deleting a or b or switching on $ab \in E_1$ does not affect H_2,

$$\begin{aligned} Q(H_1 \cup H_2) &= Q((H_1 \setminus a) \cup H_2) + Q((H_1^{ab} \setminus b) \cup H_2) \\ &= Q(H_1 \setminus a)Q(H_2) + Q(H_1^{ab} \setminus b)Q(H_2) \\ &= Q(H_1)Q(H_2), \end{aligned}$$

□

Question 17

(i) Prove that $Q(H;2) = 2^n$ for a graph H on n vertices.

(ii) Show that if H is a forest and a a leaf of H (degree 1) attached to vertex b then $Q(H;x) = Q(H \setminus a;x) + xQ(H \setminus \{a,b\};x)$.

Proposition 5.9 *For graph $H = (V,E)$ and any $U \subseteq V$ we have* $\deg Q(H) \geq \deg Q(H[U])$. *In particular,* $\deg Q(H) \geq \alpha(H)$, *where $\alpha(H)$ is the size of the largest independent (stable) set of vertices in H.*

Proof. It suffice to prove that $\deg Q(H) \geq \deg Q(H \setminus a)$ for $a \in V$.

If a is isolated then $q(H;x) = xQ(H \setminus a)$ by multiplicativity over disjoint unions. Otherwise, for $ab \in E$, $Q(H;x) = Q(H \setminus a;x) + Q(H^{ab} \setminus b;x)$, and since the interlace polynomial of a graph has nonnegative coefficients the result follows. □

Question 18

(i) Prove that, more generally than the first statement of Proposition 5.9, for a connected graph H the coefficient of x^i in $Q(H[U];x)$ is less than or equal to that of x^i in $Q(H;x)$.

(ii) Prove that $c(H)$ (number of connected components of H) is the smallest index i for which the coefficient of x^i in $Q(H;x)$ is non-zero.

Recall that for 2-in 2-out digraph D we denote by $e_k(D)$ the number of Euler cycle k-partitions of D. Let

$$e(D;x) = \sum_{k\geq 1} e_k(D)x^{k-1}.$$

Theorem 5.10 *For Euler tour C of 2-in 2-out digraph D we have*

$$e(D(C);x) = Q(H(C);x+1).$$

Proof. The proof is by induction on the number of vertices of D (number of vertices of H).

For an interlaced pair of vertices a and b in Euler tour C of D,

$$e_k(D(C)) = e_k(D(C-a)) + e_k(D(C^{ab}-a)),$$

since transposing C along a and b switches transition at a, and to each Euler cycle k-partition there corresponds one of two possible transitions at a (either given by an interlacement of one of the constituent Euler cycles, or by the transition obtained by taking the union of the two Euler cycles containing a. Euler cycle k-partitions of D associated with the one type of transition at a correspond bijectively to Euler cycle k-partitions of $D(C-a)$, while those with the other transition at a corresponds bijectively to Euler cycle k-partitions of $D(C^{ab}-a)$.). Hence,

$$\begin{aligned} e(D;x) &= e(D(C-a);x) + e(D(C^{ab}-a);x) \\ &= Q(H(C)\setminus a; x+1) + Q(H^{ab}\setminus b; x+1) \qquad \text{by induction hypothesis,} \\ &= Q(H(C); x+1). \end{aligned}$$

If there is no interlaced pair of vertices in C, then $H(C)$ has no edges and C has loop on some vertex a (see proof of Lemma 4.8). In this case, by either keeping the loop as a separate Euler cycle or gluing it to the Euler cycle passing through a, we have

$$\begin{aligned} e(D(C);x) &= xe(D(C-a);x) + e(D(C-a);x) \\ &= (x+1)e(D(C-a);x) \\ &= (x+1)Q(H(C)\setminus a; x+1) \qquad \text{by induction hypothesis,} \\ &= Q(H(C);x+1) \qquad \text{by Prop. 5.8 (multiplicativity over disjoint unions).} \end{aligned}$$

□

We finish with the relationship between the interlace polynomial and the Tutte polynomial of a plane graph. First it will be useful to describe how $e(D;x)$ behaves over connected components and blocks:

Lemma 5.11 *(i) If D_1 and D_2 are 2-in 2-out digraphs on disjoint vertex sets then*

$$e(D_1 \cup D_2; x) = xe(D_1;x)e(D_2;x).$$

(ii) If D is a 2-in 2-out digraph with cut-vertex a, C an arbitrary Euler tour of D, and D_1 and D_2 are the two connected components of the 2-in 2-out digraph $D(C-a)$, then

$$e(D_1 \cup D_2; x) = (x+1)e(D_1;x)e(D_2;x).$$

Proof. For (i) we have

$$\begin{aligned} e(D_1 \cup D_2; x) &= \sum_{k\geq 2}\left(\sum_{i+j=k} e_i(D_1)e_j(D_2)\right)x^{k-1} \\ &= xe(D_1;x)e(D_2;x). \end{aligned}$$

For (ii), if a is the cut-vertex of $D_1 \cup D_2$, there are two possibilities for a given Euler cycle k-partition of $D_1 \cup D_2$: either it has a transition at a making it the union of an Euler cycle

i-partition of D_1 and Euler cycle $(k-i)$-partition of D_1, or its transition at a joins a cycle in a Euler cycle i-partition of D_1 and a cycle in an Euler cycle $(k+1-i)$-partition of D_1. By partitioning Euler cycle k-partitions of $D_1 \cup D_2$ into these two cases, we have

$$\begin{aligned} e(D_1 \cup D_2; x) &= \sum_{k \geq 2} \left(\sum_{i+j=k} e_i(D_1) e_j(D_2) \right) x^{k-1} + \sum_{k \geq 1} \left(\sum_{i+j=k+1} e_i(D_1) e_j(D_2) \right) x^{k-1} \\ &= (x+1) e(D_1; x) e(D_2; x). \end{aligned}$$

□

Theorem 5.12 *When G is a plane graph with medial graph $\overrightarrow{m}(G)$ given the alternating orientation which orients black faces anticlockwise,*

$$e(\overrightarrow{m}(G); x) = T(G; x+1, x+1).$$

Hence if C is an Euler tour of $\overrightarrow{m}(G)$ and $H(C)$ its interlace graph, then $Q(H(C); x) = T(G; x, x)$.

Proof. We prove that $e(\overrightarrow{m}(G); x) = T(G; x+1, x+1)$ for plane graph $G = (V, E)$ by induction on the number of edges of G. The statement is true for $G = K_2$, where $m(G)$ is a single vertex with two loops and we have $e(\overrightarrow{m}(G); x) = 1 + x = T(K_2; x+1, x+1)$. (The case where $G = K_1$ is also vacuously true.)

When e is an ordinary edge of G (neither bridge nor loop),

$$\begin{aligned} T(G; x+1, x+1) &= T(G/e; x+1, x+1) + T(G \backslash e; x+1, x+1) \\ &= e(\overrightarrow{m}(G)/e; x) + e(\overrightarrow{m}(G \backslash e; x) \qquad \text{by induction hypothesis,} \\ &= e(\overrightarrow{m}(G); x), \end{aligned}$$

the last line by the fact that (cf. proof of Theorem 5.10 and Figure 14) $\overrightarrow{m}(G/e)$ and $\overrightarrow{m}(G \backslash e)$ correspond to $\overrightarrow{m}G)$ with vertex e contracted out according to the two possible transitions at e that an Euler tour may take.

If e is a bridge, then e is a cut-vertex in $\overrightarrow{m}(G)$ and

$$\begin{aligned} T(G; x+1, x+1) &= (x+1) T(G/e; x+1, x+1) \\ &= (x+1) T(G_1; x+1, x+1) T(G_2; x+1, x+1) \quad \text{blocks } G_1 \text{ and } G_2 \text{ of } G/e, \\ &= (x+1) e(\overrightarrow{m}(G_1); x) e(\overrightarrow{m}(G_2); x) \qquad \text{by induction hypothesis,} \\ &= e(\overrightarrow{m}(G); x) \qquad \text{cut-vertex } e \text{ of } m(G), \text{ Lemma 5.11(ii).} \end{aligned}$$

If e is a loop, then in $\overrightarrow{m}(G)$ the vertex e is a cut-vertex with a loop. Let $\overrightarrow{m}(G)'$ denote the digraph obtained by contracting out the vertex e (see Figure 15). Then $\overrightarrow{m}(G)' = \overrightarrow{m}(G \backslash e)$ and we have

$$\begin{aligned} T(G; x+1, x+1) &= (x+1) T(G \backslash e; x+1, x+1) \\ &= (x+1) e(\overrightarrow{m}(G)'; x) \qquad \text{by induction hypothesis,} \\ &= e(\overrightarrow{m}(G); x) \qquad \text{cut-vertex } e \text{ of } m(G), \text{ Lemma 5.11(ii).} \end{aligned}$$

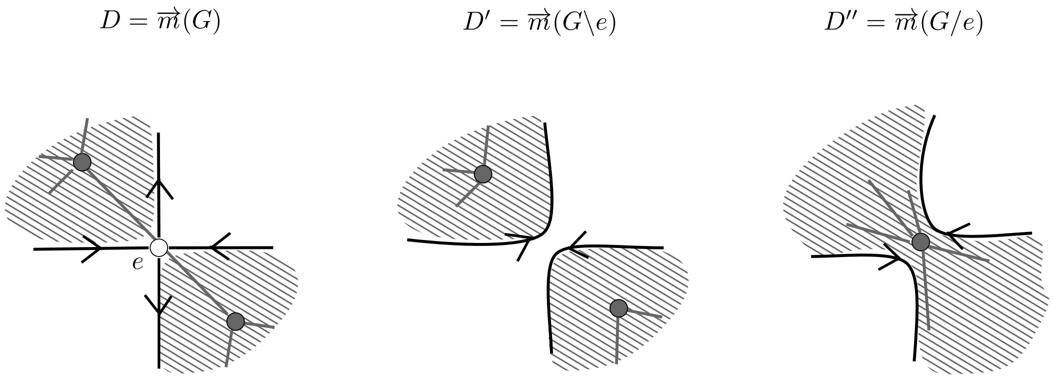


Figure 14: The two types of transition at a vertex of a plane 2-in 2-out digraph D, equal to $m(G)$ for some plane graph G, with alternating orientation anticlockwise around black faces. The black transition corresponds to edge deletion in G, and the white transition to edge contraction.

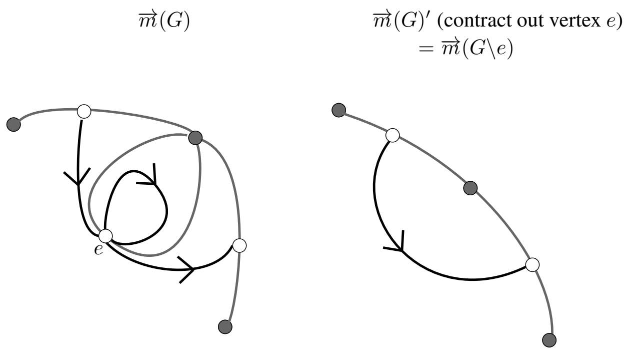


Figure 15: Contracting out a vertex with a loop in the oriented medial graph (proof of Theorem 5.12).

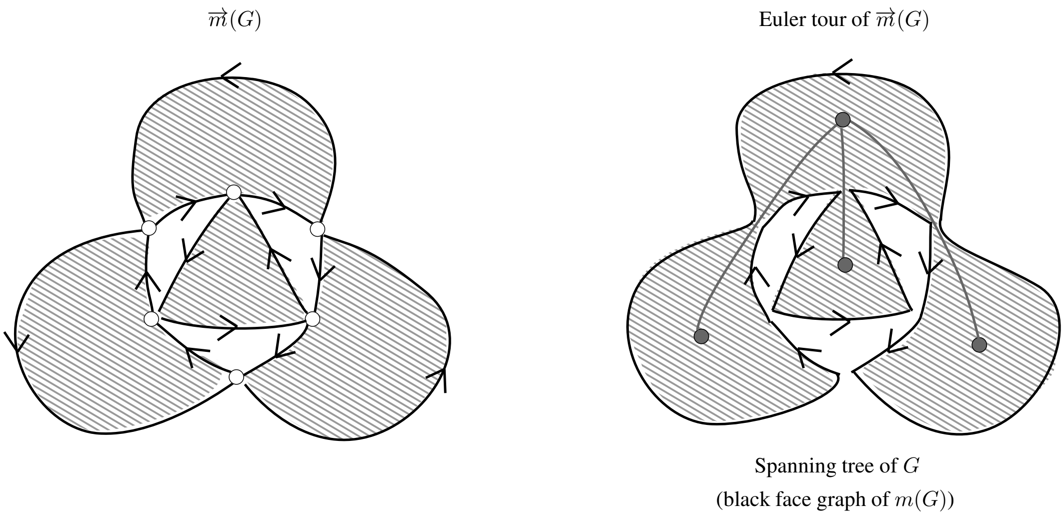


Figure 16: Correspondence between Euler tours of an alternating orientation of the medial graph of a plane graph G and spanning trees of G. A tour of the medial graph forms a simple closed curve in the plane unifying all the black faces into one region.

(The last line can also be seen by considering the two cases where the loop is separate from the other Euler cycles comprising the Euler cycle partition, or joined to an existing Euler cycle.) □

Corollary 5.13 *If G is a connected plane graph then the number of Euler tours of $\overrightarrow{m}(G)$ is equal to the number of spanning trees of G.*

Question 19 Prove Corollary 5.13 directly. (See Figure 16. Also [10].)

Given a 2-in 2-out digraph D there is a unique *anticycle partition* formed by following two out-edges then two in-edges, and repeating this until all edges have been traversed (once an edge is encountered again this closes a component of the anticycle, which is a cycle which when traversed alternates in the orientation of its edges forward and backward). The number of components in this anticycle partition is denoted by $a(D)$. For the example $D = \overrightarrow{m}(K_4)$ see Figure 17, which interprets this anticycle partition as the diagram of a link (in this case three unknots linked as Borromean rings). We shall develop this connection in the next section about the Kauffman bracket and Jones polynomial.

The following evaluation of the interlace polynomial of the interlace graph of a 2-in 2-out digraph D gives another interpretation of the Tutte polynomial evaluation $T(G; -1, -1)$ when $D = \overrightarrow{m}(G)$ for plane G:

Theorem 5.14 *Let D be a 2-in 2-out digraph on n vertices with Euler tour C and $a(D)$ the number of components in its anticycle partition. Then*

$$e(D;-2) = (-1)^n(-2)^{a(D)-1}.$$

PROOF. The proof is by induction on n. When $n = 1$ the digraph D is a single vertex with two loops and we have $a(D) = 1$ and $e(D;x) = 1+x$, so $e(D;-2) = -1 = (-1)(-2)^0$ and the base case is true.

Consider D on $n > 1$ vertices and a vertex a of D. Let D'_a denote the digraph D with a contracted out according to one possible transition for Euler cycles at a and D''_a the digraph D with a contracted out according to the other possible transition for Euler cycles at a. (Given an Euler tour C interlaced at a and b we can take $D'_a = D(C-a)$ and $D''_a = D(C^{ab} - a)$.)

If the vertex a belongs to two anticycles then $a(D'_a) = a(D''_a) = a(D) - 1$ and

$$\begin{aligned} e(D;-2) &= e(D'_a;-2) + e(D''_a;-2) \\ &= (-1)^{n-1}(-2)^{a(D)-2} + (-1)^{n-1}(-2)^{a(D)-2} \qquad \text{by induction hypothesis,} \\ &= (-1)^n(-2)^{a(D)-1}. \end{aligned}$$

If on the other hand the vertex a belongs to a single anticycle component then $\{a(D'_a), a(D''_a)\} = \{a(D), a(D)+1\}$ and

$$\begin{aligned} e(D;-2) &= (-1)^{n-1}(-2)^{a(D)-1} + (-1)^{n-1}(-2)^{a(D)} \\ &= (-1)^n(-2)^{a(D)-1}. \end{aligned}$$

□

Corollary 5.15 *If G is a plane graph and $\overrightarrow{m}(G)$ its medial graph with alternating orientation then*

$$T(G;-1,-1) = (-1)^{|E(G)|}(-2)^{a(\overrightarrow{m}(G))-1},$$

where $a(\overrightarrow{m}(G))$ is the number of components in the anticycle partition of $\overrightarrow{m}(G)$.

Actually, for a general graph G we have the evaluation

$$T(G;-1,-1) = (-1)^{|E(G)|}(-2)^{\dim(\mathcal{C}\cap\mathcal{C}^\perp)},$$

where $\mathcal{C}$ is the (binary) cycle space of G and $\mathcal{C}^\perp$ the cutset space of G. The space $\mathcal{C}\cap\mathcal{C}^\perp$ is called the *bicycle space* of G. See [8] for an elucidation of the correspondence.

6 The Kauffman bracket of a link

[See handwritten lecture notes.]

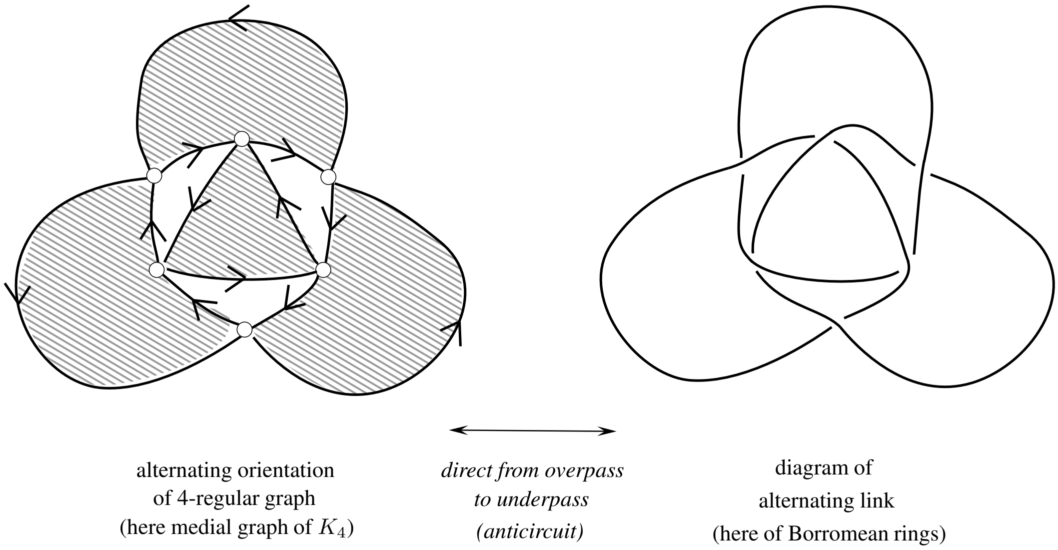


Figure 17: Canonical alternating orientation of $m(K_4)$ and interpretation of the anticycle partition (into three components) as the diagram of an alternating link, which in this case represents the Borromean rings.

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