Vybrané kapitoly z kombinatoriky: Many facets of the Tutte polynomial

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1 Isomorphism. Graph invariants. 2-isomorphism.

For a basic introduction to graphs and isomorphism see e.g. [35, ch. 4].

For us a graph $G = (V, E)$ may have parallel edges and loops. (Usually called multigraphs or pseudographs. Biggs [4] calls them general graphs, and simple graphs he calls strict graphs.) The sets V and E come with a mapping $E \to V \cup \binom{V}{2}$ that assigns to each edge e its endpoint(s).

A *simple graph* has no multiple edges or loops; here an edge e can be identified with an unordered pair of vertices uv (its endpoints u and v).

Definition 1.1. *Two simple graphs $G = (V, E)$ and $G' = (V', E')$ are* isomorphic *if there is a bijection $f : V \to V'$ such that $uv \in E$ if and only if $f(u)f(v) \in E'$, for all $u, v \in V$.*

The problem of testing whether a pair of graphs are isomorphic is in NP (non-deterministic polynomial time): a polynomial-time checkable certificate for isomorphism is a suitable bijection f. The problem joins integer factorization (given n and $1 < m < n$, does n have a factor d with $1 < d \leq m$?) in being one of the few NP problems not known either to lie in P (polynomial time) or to be NP-complete (as difficult as any NP problem you care to name, such as deciding whether a graph has a proper 3-colouring).

Two multigraphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there are functions $f_0 : V \to V'$ and $f_1 : E \to E'$ such that

(i) if e has endpoints u and v then $f_1(e)$ has endpoints $f_0(u)$ and $f_0(v)$;

(ii) f_0 and f_1 are bijections.

In other words, an isomorphism between multigraphs is an isomorphism between their underlying simple graphs together with the condition that edge multiplicities are the same (including loops).

A multigraph $G = (V, E)$ can be represented by its adjacency matrix $A = A(G)$ with (u, v) entry equal to the number of edges joining u and v. Multigraphs G and G' are isomorphic if and only if the matrices $A(G)$ and $A(G')$ are permutation-equivalent.

Definition 1.2. *A* graph invariant *is a function on graphs such that $f(G) = f(G')$ whenever G and G' are isomorphic.*

Informally, a graph invariant ignores vertex labels and any embedding or other representation of the graph. A graph invariant may be numerical, polynomial, structural (e.g. cycle matroid, basis exchange graph), etc.

Examples of graph invariants: order and size, number of connected components, connectivity, degree sequence (i.e., non-increasing sequence of vertex degrees – if the sequence depended on vertex labels then it would not be a graph invariant), spectrum (i.e., set of eigenvalues of adjacency matrix), genus (least genus of orientable surface in which it can be embedded), independence number, chromatic number, whether the graph is Hamiltonian, ...

Functions on graphs that change their value under some isomorphism include any function that depends on vertex labels (its value changing for some graph with some permutation of the labels). For planar graphs the face structure in an embedding of G in the sphere is also not a graph invariant. The faces of G are the vertices of the geometric dual G^*: embeddings are equivalent if the geometric duals are isomorphic. Some planar graphs have inequivalent embeddings. (The geometric dual is an operation on *embedded graphs* not just graphs.) Likewise, the number of crossings in a drawing of a graph G in the plane is not a graph invariant. On the other hand, the crossing number $\mathrm{cr}(G)$, defined as the least number of crossings in an embedding of G in the plane, is a graph invariant.

For more on planar graphs and their duals see e.g. [35, ch. 6], [10, ch. 4]. A 3-connected planar graph has a unique embedding in the plane (Whitney, 1932).

Exercise 1.3. *Give an example of a 2-connected planar graph G with two inequivalent embeddings, i.e., find G with two embeddings with non-isomorphic geometric duals G^*. For any given k, describe a 2-connected planar graph with at least k pairwise inequivalent embeddings.*

Can you find a condition for a 2-connected planar graph to have a unique embedding in the plane?

Definition 1.4. *Two graphs G and G' are* 2-isomorphic *if G can be transformed into G' by means of the following two operations and their inverses:*

(i) Identify two vertices in different connected components of G.

(ii) Suppose G is obtained from disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2, and identifying v_1 of G_1 and v_2 of G_2. The Whitney twist *of G is the graph obtained by identifying u_1 with v_2 and u_2 with v_1.*

The first operation joins two components in a 1-cut (its inverse separating a graph at a 1-cut). The Whitney twist acts by flipping the graph G about one of its 2-cuts.

Proposition 1.5. *If two graphs are 2-isomorphic then the structure of their cycles are the same (more precisely, their cycle matroids are isomorphic).*

Proof. Clearly the edge set of cycles have the same structure when identifying two vertices in different components. Suppose G' is obtained from G by a Whitney twist about a give 2-cut of G. A cycle that does not pass through either vertex of the 2-cut remains unchanged. A cycle of G passing through one of the vertices of the 2-cut must pass through the other. If traversing this cycle we encounter the edges $e_1, e_2, \ldots, e_i, f_1, f_2, \ldots, f_j$, where the e-edges belong to G_1 and the f-edges to G_2, then in the Whitney twist corresponds the cycle in whose traversal we meet the edges in the order $e_1, \ldots, e_i, f_j, f_{j-1}, \ldots, f_1$. Thus the edge sets of cycles are the same in both graphs. □

Theorem 1.6. Whitney [53] *The cycle matroids of G and G' are isomorphic if and only if G and G' are 2-isomorphic. In particular, if G is 3-connected and G has isomorphic cycle matroid to G' then G and G' are isomorphic.*

Geometric duals of a plane graph are 2-isomorphic, even though they may not be isomorphic when the graph is not 3-connected.

2 The chromatic polynomial.

For graph colouring see e.g. [5, ch. V], [10, ch. 5]. For the chromatic polynomial see e.g. [4, ch. 9] and [11].

Definition 2.1. *A* colour-partition *of a graph $G = (V, E)$ is a partition of V into disjoint non-empty subsets, $V = V_1 \cup V_2 \cup \cdots \cup V_k$, such that the colour-class V_i is an independent set of vertices in G, for each $1 \leq i \leq k$ (i.e., each induced subgraph $G[V_i]$ has no edges).*

The chromatic number $\chi(G)$ *is the least natural number k for which such a partition is possible.*

If G has a loop then it has no colour-partitions. Adding or removing edges in parallel to a given edge makes no difference to what counts as a colour-partition, since its definition depends only on whether vertices are adjacent or not.

Let $z^{\underline{i}} = z(z-1)\cdots(z-i+1)$.

Definition 2.2. *Let $m_i(G)$ denote the number of colour partitions of G into i colour-classes. The* chromatic polynomial *of G is defined by*

$$P(G; z) = \sum_{i=1}^{|V|} m_i(G) z^{\underline{i}}.$$

Definition 2.3. *A* proper k-colouring *of the vertices of $G = (V, E)$ is a function $f : V \to [k]$ with the property that $f(u) \neq f(v)$ whenever $uv \in E$.*

Note that the vertices of a graph are regarded as labelled and colours are distinguished: colourings are different even if equivalent up to an automorphism of G or a permutation of the colour set.

Exercise 2.4. *Prove that if G and G' can be obtained one from the other by a sequence of Whitney twists then $P(G; z) = P(G'; z)$. More generally, if G and G' are 2-isomorphic then $z^{-c(G)}P(G; z) = z^{-c(G')}P(G'; z)$, where $c(G)$ denotes the number of connected components of G.*

Proposition 2.5. *If $k \in \mathbb{N}$ then $P(G; k)$ is the number of proper vertex k-colourings of G.*

Proof. To every proper colouring in which exactly i colours are used there corresponds a colour partition into i colour classes. Conversely, given a colour partition into i classes there are $k^{\underline{i}}$ ways to assign colours to them. Hence the number of proper k-colourings is $\sum m_i(G) k^{\underline{i}} = P(G; k)$. □

For example, when G is the complete graph on n vertices,

$$P(K_n; z) = z^{\underline{n}} = z(z-1)\cdots(z-n+1),$$

with $m_i(K_n) = 0$ for $1 \leq i \leq n-1$ and $m_n(K_n) = 1$.

In general, if G has n vertices then $m_n(G) = 1$ so that $P(G; z)$ has leading coefficient 1. The constant term $P(G; 0)$ is zero since z is a factor of $z^{\underline{i}}$ for each $1 \leq i \leq n$. If E is non-empty then $P(G; 1) = 0$, so that $z - 1$ is a factor of $P(G; z)$. More generally, the integers $0, 1, \ldots, \chi(G) - 1$ are all roots of $P(G; z)$, and $\chi(G)$ is the first positive integer that is not a root of $P(G; z)$.

Proposition 2.6. *If $G = (V, E)$ is a simple graph on n vertices and m edges then the coefficient of z^{n-1} in $P(G; z)$ is equal to $-m$.*

Proof. A partition of n vertices into $n-1$ subsets necessarily consists of $n-2$ singletons and one pair of vertices $\{u, v\}$. This is a colour-partition if and only if $uv \notin E$. Hence $m_{n-1}(G) = \binom{n}{2} - m$, where m is the number of pairs of adjacent vertices, equal to the number of edges of G when there are no parallel edges. Then

$$[z^{n-1}]P(G; z) = -(1 + 2 + \cdots + n-1)m_n(G) + m_{n-1}(G) = -m.$$

$\square$

Proposition 2.7. *If G is the disjoint union of G_1 and G_2 then $P(G; z) = P(G_1; z)P(G_2; z)$.*

Exercise 2.8. *Prove that*

$$P(G; z_1 + z_2) = \sum_{V_1 \uplus V_2 = V} P(G[V_1]; z_1)P(G[V_2]; z_2).$$

(This makes the chromatic polynomial of a graph a set function of binomial type in the terminology of [54].)

Proposition 2.9. *Suppose G' is obtained from G by joining a new vertex to each vertex of an r-clique in G. Then $P(G'; z) = (z - r)P(G; z)$.*

Proof. The identity holds when z is equal to a positive integer k, for to each proper k-colouring of G there are exactly $k - r$ colours available for the new vertex to extend to a proper colouring of G'. $\square$

Consequently, if G is a tree on n vertices then $P(G; z) = z(z-1)^{n-1}$ (every tree on $n \geq 2$ vertices has a vertex of degree 1 attached to a 1-clique in a tree on $n-1$ vertices).

A *chordal graph* is a graph such that every cycle of length four or more contains a chord, i.e., there are no induced cycles of length four or more. A chordal graph can be constructed by successively adding a new vertex and joining it to a clique of the existing graph (Dirac, 1961). By Proposition 2.9, for a chordal graph G we have $P(G; z) = z^{c(G)}(z-1)^{k_1} \cdots (z-s)^{k_s}$, where $k_1 + \cdots + k_s = |V| - c(G)$ and $s = \chi(G) - 1$.

2.1 Deletion and contraction

The graph $G \backslash e$ obtained from $G = (V, E)$ by deleting an edge e has vertex set V and edge set $E \setminus \{e\}$.

The graph G/e obtained from $G = (V, E)$ by contracting an edge $e = uv$ has vertex set $V \setminus \{u, v\} \cup \{w\}$ and edge set $E \setminus \{uv\} \cup \{xw : xu \in E\} \cup \{yw : yv \in E\}$, where w is the vertex obtained by identifying u and v. (Identify vertices u and v and remove the loop that e becomes.)

$G \backslash e$ has one edge fewer than G. If e is a bridge (i.e., a cut-edge) then $G \setminus e$ has one more connected component than G.

If e is not a loop then G/e has one edge and vertex fewer than G. If e is a loop then $G/e \cong G \setminus e$. The *rank* of G is defined by $r(G) = |V(G)| - c(G)$. The rank of a connected graph is equal to the number of edges in a spanning tree of G, that of a disconnected graph the size of a maximal spanning forest. Note that

$$r(G\backslash e) = \begin{cases} r(G) & e \text{ not a bridge,} \\ r(G) - 1 & e \text{ a bridge,} \end{cases}$$

and

$$r(G/e) = \begin{cases} r(G) - 1 & e \text{ not a loop,} \\ r(G) & e \text{ a loop.} \end{cases}$$

A *minor* of G is a graph that can be obtained from G by a sequence of edge deletions, edge contractions and vertex deletions. Denoting the graph obtained by deleting a vertex v by $G - v$,

$$r(G - v) = \begin{cases} r(G) & v \text{ an isolated vertex (possibly with loops),} \\ r(G) - 1 & \text{otherwise.} \end{cases}$$

Thus if H is a minor of G then $r(H) \leq r(G)$.

Proposition 2.10. *The chromatic polynomial satisfies the relation*

$$P(G; z) = P(G\backslash e; z) - P(G/e; z).$$

Proof. When e is a loop we have $P(G; z) = 0 = P(G\backslash e; z) - P(G/e; z)$ since $G\backslash e \cong G/e$. When e is parallel to another edge of G we have $P(G; z) = P(G\backslash e; z)$ and $P(G/e; z) = 0$ since G/e has a loop.

Suppose then that e is not a loop or parallel to another edge. Consider the proper vertex k-colourings of $G\backslash e$. Those which colour the ends of e differently are in bijective correspondence with proper k-colourings of G, while those that colour the ends the same are in bijective correspondence with proper k-colourings of G/e. Hence $P(G\backslash e; k) = P(G; k) + P(G/e; k)$ for each positive integer k. □

We can use this recurrence to dismantle a sparse connected graph into empty graphs (or stop at trees since we know that $P(G; z) = z(z-1)^{n-1}$ for a tree on n vertices). A binary deletion-contraction tree of depth $|E(G)|$ is required to reach cocliques at the leaves. When multiple edges appear they can be deleted to leave simple edges (in other words, contraction of an edge parallel to another edge gives a loop and this contributes zero to the chromatic polynomial). The leaf cocliques $\overline{K}_i$ each contribute z^i to the chromatic polynomial of G.

For a connected simple graph, a binary deletion–contraction tree of depth $|E(G)| - |V(G)| + 1$ has trees at its leaves, where at each level a non-bridge e is chosen to preserve connectedness, or a multiple edge is just deleted (the contraction giving a loop). The number of contractions performed to reach a tree of i edges at a leaf is equal to $|V(G)| - 1 - i$, since only contraction reduces the number of vertices and so to get from G with $|V(G)|$ vertices to a tree on $i + 1$ vertices requires this many contractions. Recall that any branches of the tree with loops "die off" since the contribution to the chromatic polynomial of G is zero along such a branch.

Writing the recurrence in the form $P(G\backslash e; z) = P(G; z) + P(G/e; z)$, we can "fill out" a dense connected graph to complete graphs. Add the edge e to $G\backslash e$ to make G, and if G/e has parallel edges these can be removed without affecting the value of $P(G/e; z)$: in any event, the number of non-edges in both G and (the simplified graph) G/e is one less than in $G\backslash e$. Hence, starting with a simple connected graph G, $\binom{|V(G)|}{2} - |E(G)|$ deletion–contraction steps are required to reach complete graphs.

See [11, ch. 1] for an illustration of deletion–contraction computation trees for the chromatic polynomial.

The chromatic polynomial of graphs in the families $\{K_n : n \in \mathbb{N}\}$ and $\{\overline{K}_n : n \in \mathbb{N}\}$ give respectively the bases $\{z^{\underline{n}}\}$ and $\{z^n\}$ for polynomials in z. Recall that

$$z^{\underline{n}} = \sum_{k=1}^{n} s(n,k) z^k,$$

where $s(n,k)$ are the signed Stirling numbers of the first kind, defined recursively by

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k),$$

$$\begin{cases} s(r,0) = 0 & r = 1, 2, ... \\ s(r,r) = 1 & r = 0, 1, 2, ... \end{cases}.$$

(The number $(-1)^{n-k} s(n,k)$ counts the number of permutations of an n-set with exactly k cycles.) Inversely,

$$z^n = \sum_{k=1}^{n} S(n,k) z^{\underline{k}},$$

where the $S(n,k)$ are the Stirling numbers of the second kind, counting the number of partitions of an n-set into k non-empty sets. (Indeed, we know that $P(\overline{K}_n; z) = \sum m_i(\overline{K}_n) z^{\underline{i}}$, where $m_i(\overline{K}_n)$ is the number of partitions of the n isolated vertices of $\overline{K}_n$ into i independent subsets.)

Likewise, the bases $\{z^n\}$ and $\{z(z-1)^n\}$ can be transformed into each other by the relations

$$z^n = \sum_{k=0}^{n-1} \binom{n-1}{k} z(z-1)^k,$$

and

$$z(z-1)^n = \sum_{k=1}^{n+1} (-1)^{n-k+1} \binom{n}{k-1} z^k.$$

Deletion–contraction can be used to give another proof of the formula for the chromatic polynomial of trees:

Proposition 2.11. *The chromatic polynomial of a tree on n vertices is given by $z(z-1)^{n-1}$.*

Proof. Any tree on $n \geq 2$ vertices has a vertex of degree 1. Assume inductively that the result is true for all trees on at most $n-1$ vertices (base case of isolated vertex has $P(K_1; z) = z$). Given a tree T on n vertices, suppose e is an edge with an endpoint of degree 1. Then $T \backslash e$ has two components, one an isolated vertex, the other isomorphic to T/e. Hence $P(T \backslash e; z) = zP(T/e; z)$, so that by the deletion–contraction recurrence we have $P(T; z) = (z-1)P(T/e; z) = (z-1)z(z-1)^{n-2}$. □

Relative to the basis $\{z(z-1)^n\}$ for polynomials in z, for a connected graph G we have

$$P(G; z) = \sum_{i=0}^{|V|-1} (-1)^{|V|-1-i} t_{i,0}(G) z(z-1)^i, \tag{1}$$

where $t_{i,0}(G)$ has a combinatorial interpretation in terms of spanning trees of G (namely, spanning trees of external activity 0 and *internal activity* i: see later when defining the Tutte polynomial). For connected G the $t_{i,0}(G)$ are positive for $1 \le i \le |V|-1$: this follows from the same deletion–contraction argument given in the proof of Proposition 2.13 below. When G has at least one edge we have $t_{0,0}(G) = 0$. Also $t_{|V|-1,0}(G) = 1$. The number $t_{i,0}(G)$ is equal to the number of trees of size i at the end of the deletion–contraction computation tree, where a branch terminates as soon as a tree is obtained (after removal of any parallel edges). In particular, no matter in which order edges are deleted and contracted the distribution of trees at the leaves by order is the same [13]. As a consequence of the expansion (1), the chromatic polynomial of a connected graph G is non-zero with sign $(-1)^{|V|-1}$ for $z \in (0,1)$.

Let $z^{\overline{k}}$ denote the rising factorial $z(z+1)\cdots(z+k-1)$. Brenti [6] proved that

$$P(G;z) = \sum_{i=1}^{|V|} (-1)^{|V|-i} c_i(G) z^{\overline{i}},$$

where $c_i(G)$ is the number of set partitions $V_1 \cup V_2 \cup \cdots \cup V_i$ of V into i blocks paired with an acyclic orientation of $G[V_1] \cup G[V_2] \cup \cdots \cup G[V_i]$.

See [54] for expressions for the coefficients of the chromatic polynomial relative to any polynomial basis $\{b_i(z)\}$ of binomial type (meaning it satisfies $b_j(x+y) = \sum_{i=0}^{j} \binom{j}{i} b_i(x) b_{j-i}(y)$).

Exercise 2.12. *Use the deletion–contraction recurrence to find the chromatic polynomial of the cycle C_n.*

Proposition 2.13. *If*

$$P(G;z) = \sum_{i=0}^{|V|} (-1)^i b_i(G) z^{|V|-i},$$

then $b_i(G) > 0$ for $0 \le i \le r(G)$, and $b_i(G) = 0$ for $r(G) < i \le |V|$.

Proof. We shall show that

$$(-1)^{|V|} P(G;-z) = \sum_{i=0}^{r(G)} b_i(G) z^{|V|-i}$$

has strictly positive coefficients. We assume G has no loops, for in this case $P(G;z) = 0$. By the deletion–contraction formula, and using the fact that $|V(G\backslash e)| = |V(G)|$ and $|V(G/e)| = |V(G)| - 1$ when e is not a loop,

$$(-1)^{|V(G)|} P(G;-z) = (-1)^{|V(G\backslash e)|} P(G\backslash e;-z) + (-1)^{|V(G/e)|} P(G/e;-z).$$

Hence

$$b_i(G) = b_i(G\backslash e) + b_{i-1}(G/e).$$

Assume inductively on the number of edges that $b_i(G) > 0$ for $0 \le i \le r(G)$, and that $b_i(G) = 0$ otherwise. As a base for induction, $(-1)^n P(\overline{K}_n; -z) = z^n$.

By inductive hypothesis, for $0 \le i \le r(G\backslash e)$ we have $b_i(G\backslash e) > 0$ and for $0 \le i-1 \le r(G/e)$ we have $b_{i-1}(G/e) > 0$. When e is not a bridge $r(G\backslash e) = r(G)$ and so $b_i(G\backslash e) > 0$ for $0 \le i \le r(G)$, otherwise for a bridge $r(G\backslash e) = r(G) - 1$ and in this case $b_i(G) > 0$ for $0 \le i \le r(G) - 1$. Since e is not a loop $r(G/e) = r(G) - 1$, so we have $b_{i-1}(G/e) > 0$ for $1 \le i \le r(G)$. Together these inequalities imply $b_i(G) > 0$ for $0 \le i \le r(G)$.

Clearly z divides $P(G;z)$ for a connected graph. It follows that $z^{c(G)}$ is a factor of $P(G;z)$ by multiplicativity of the chromatic polynomial over disjoint unions. Hence $b_i(G) = 0$ for $r(G) < i \le |V(G)|$. Also, the degree of $P(G;z)$ is $|V(G)|$ by its definition, so there are no remianing non-zero coefficients. ☐

We shall see below in Whitney's Broken Cycle Theorem that the numbers $b_i(G)$ have a combinatorial interpretation in terms of forests of G. For now we use the property that the coefficients alternate in sign to deduce a result about roots of the chromatic polynomial.

Proposition 2.14. *The only rational roots of* $P(G;z)$ *are* $0, 1, \ldots, \chi(G)-1$.

Proof. If G has a proper k-colouring then it has a proper $(k+1)$-colouring. Hence $\chi(G)+i$ is not a root for non-negative integer i. Since the coefficients of $P(G;z)$ alternate in sign there are no negative roots. Since $P(G;z)$ is a monic polynomial over $\mathbb{Z}$ it has no non-integer rational roots. □

The root 0 has multiplicity $c(G)$, the root 1 multiplicity the number of blocks of G. By definition, only non-bipartite graphs have 2 as a root of their chromatic polynomial. For planar graphs 4 is not a root by the Four Colour Theorem. Tutte in 1970 observed that for planar graphs there is often a zero close to τ^2 where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio, and proved that if G is a triangulation of the plane then $P(G;\tau^2) \leq \tau^{5-n}$. See e.g. [11, ch. 12-14] and [29] for more about chromatic roots.

Proposition 2.15. *For a simple graph* G *on* n *vertices and* m *edges the coefficient of* z^{n-2} *in* $P(G;z)$ *is equal to* $\binom{m}{2} - t$, *where* t *is the number of triangles in* G.

Proof. The assertion is true when $m = 0, 1, 2$. Suppose G has n vertices and $m \geq 3$ edges. For a non-loop e, $b_2(G) = b_2(G\backslash e) - b_1(G/e)$. Inductively, $b_2(G\backslash e) = \binom{m-1}{2} - t_0$, where t_0 is the number of triangles in G not containing the edge e, the graph $G\backslash e$ being simple. In a triangle $\{e, e_1, e_2\}$ of G containing e, the edges e_1, e_2 do not appear in any other triangle of G containing e, since G is simple. When e is contracted the edges e_1 and e_2 become parallel edges in G/e, and moreover there are no other edge parallel to these. Hence for each triangle $\{e, e_1, e_2\}$ of G we remove one parallel edge in G/e in order to reduce it to a simple graph. So $b_1(G/e) = (m-1) - t_1$, where t_1 is the number of triangles of G containing e. With $t_0 + t_1 = t$ equal to the number of triangles in G, the result now follows by induction. □

Proposition 2.16. *If* $P(G;z) = z(z-1)^{n-1}$ *then* G *is a tree on* n *vertices, and more generally* $P(G;z) = z^c(z-1)^{n-c}$ *implies* G *is a forest on* n *vertices with* c *components.*

Proof. The degree of $P(G;z)$ is n so G has n vertices. The coefficient of z^c is non-zero but z^{c-1} has zero coefficient, hence by Proposition 2.13 G has c connected components. Finally, reading off the coefficient of z^{n-c-1} tells us that the number of edges is $n-c$, so that G is a forest on n vertices with c components. □

Exercise 2.17. *Prove that if* $P(G;z) = P(K_n;z)$ *then* $G \cong K_n$ *and that if* $P(G;z) = P(C_n;z)$ *then* $G \cong C_n$.

2.2 Joins and clique-sums.

The *join* $G_1 + G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \cup V_2$ and edge set

$$E_1 \cup E_2 \cup \{xy : x \in V_1, y \in V_2\}.$$

Proposition 2.18. *The chromatic polynomial of the join $G_1 + G_2$ is given by*

$$P(G_1 + G_2; z) = P(G_1; z) \circ P(G_2; z),$$

where the $\circ$ operation is defined by $z^{\underline{i}} \circ z^{\underline{j}} = z^{\underline{i+j}}$, extended linearly to polynomials.

Proof. The number of colour-partitions of $G = G_1 + G_2$ is given by

$$m_k(G) = \sum_{i+j=k} m_i(G_1)m_j(G_2),$$

since every vertex of G_1 is adjacent in G to every vertex of G_2, so that any colour-class of vertices in G is either a colour class of G_1 or a colour class of G_2. □

Exercise 2.19. *Find the chromatic polynomial of the wheel W_n on $n+1$ vertices.*

Exercise 2.20. *Find an expression for the chromatic polynomial of the complete bipartite graph $K_{r,s}$ relative to the factorial basis $\{z^{\underline{n}}\}$.*

Suppose graphs G_1 and G_2 both contain a clique of order k. The *clique-sum* of G_1 and G_2 is formed by identifying the vertices of the given k-clique in the disjoint union of G_1 and G_2. The inverse notion is quasi-separation:

Definition 2.21. *A connected graph $G = (V, E)$ is* quasi-separable *if there is a subset U of V such that $G[U]$ is a complete graph and $G[V \setminus U]$ is disconnected. G is* separable *if $|U| \le 1$.*

In a quasi-separable graph we have $V = V_1 \cup V_2$ with $G[V_1 \cap V_2]$ complete and no edges joining $V_1 \setminus (V_1 \cap V_2)$ to $V_2 \setminus (V_1 \cap V_2)$. The pair (V_1, V_2) is called a *quasi-separation* of G.

Proposition 2.22. *If G has quasi-separation (V_1, V_2) then*

$$P(G; z) = \frac{P(G[V_1]; z)P(G[V_2]; z)}{P(G[V_1 \cap V_2]; z)}.$$

In particular, if G is a connected graph with 2-connected blocks $G_1, \ldots, G_k$ then

$$P(G; z) = z^{1-k} P(G_1; z)P(G_2; z) \cdots P(G_k; z).$$

Proof. It suffices to prove the identity when z is a positive integer k. Show that each proper colouring of the clique $G[V_1 \cap V_2]$ extends to $P(G[V_1]; k)/P(G[V_1 \cap V_2]; k)$ proper colourings of $G([V_1])$, and independently to $P(G[V_2]; k)/P(G[V_1 \cap V_2]; k)$ proper colourings of $G([V_2])$. Seeing that such a proper colouring of the clique $G[V_1 \cap V_2]$ also extends to $P(G; k)/P(G[V_1 \cap V_2]; k)$ proper colourings of G, we have

$$\frac{P(G; k)}{P(G[V_1 \cap V_2]; k)} = \frac{P(G[V_1]; k)}{P(G[V_1 \cap V_2]; k)} \frac{P(G[V_2]; k)}{P(G[V_1 \cap V_2]; k)}.$$

□

2.3 Subgraph expansions.

Theorem 2.23. *The chromatic polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$P(G; z) = \sum_{A \subseteq E} (-1)^{|A|} z^{c(A)},$$

where $c(A)$ is the number of connected components in the spanning subgraph (V, A).

Proof. We prove the identity when z is a positive integer k.

For an edge $e = uv$ let $M_e = \{f : V \to [k] : f(u) = f(v)\}$. Then

$$\bigcap_{e \in E} \overline{M}_e = \{f : V \to [k] : \forall_{uv \in E}\ f(u) \neq f(v)\}$$

is the set of proper k-colourings of G. By the principle of inclusion–exclusion,

$$\left| \bigcap_{e \in E} \overline{M}_e \right| = \sum_{A \subseteq E} (-1)^{|A|} \left| \bigcap_{a \in A} M_a \right|.$$

But $\left|\bigcap_{a \in A} M_a\right| = k^{c(A)}$, since a function $f : V \to [k]$ monochrome on each edge of A is constant on each connected component of (V, A), and conversely assigning each connected component a colour independently yields such a function f. $\square$

In this subgraph expansion there are many cancellations. If $e \in A$ belongs to a cycle of (V, A) then the sets A and $A \setminus \{e\}$ have contributions to the sum that cancel. Whitney's Broken Cycle expansion results by pairing off subgraphs in a systematic way.

Let $G = (V, E)$ be a simple graph whose edge set has been ordered $e_1 < e_2 < \cdots < e_m$. A *broken cycle* is the result of removing the first edge from some cycle, i.e., a subset $B \subseteq E$ such that for some edge e_l the edges $B \cup \{e_l\}$ form a cycle in G and $i > l$ for each $e_i \in B$.

Theorem 2.24. *Let G be a simple graph on n vertices with edges totally ordered, and let $P(G; z) = \sum (-1)^i b_i(G) z^{n-i}$. Then $b_i(G)$ is the number of subgraphs of G which have i edges and contain no broken cycles.*

Proof. Suppose $B_1, \ldots, B_t$ is a list of the broken cycles in lexicographic order based on the ordering of E. Let f_j $(1 \leq j \leq t)$ denote the edge which when added to B_j completes a cycle. Note that $f_j \notin B_k$ when $k \geq j$ (otherwise B_k would contain in f_j an edge smaller than any edge in B_j, contrary to lexicographic ordering).

Define $\mathcal{S}_0$ to be the set of subgraphs of G containing no broken cycle and for $1 \leq j \leq t$ define $\mathcal{S}_j$ to be the set of subgraphs containing B_j but not B_k for $k > j$. Then $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_t$ is a partition of the set of all subgraphs of G.

If $A \subseteq E$ does not contain f_j, then A contains B_j if and only if $A \cup \{f_j\}$ contains B_j. Further, A contains B_k $(k > j)$ if and only if $A \cup \{f_j\}$ contains B_k, since f_j is not in B_k either. If one the subgraphs A and $A \cup \{f_j\}$ are in $\mathcal{S}_j$ then both are, and since $c(A) = c(A \cup \{f_j\})$ the contributions to the alternating sum cancel.

The only terms remaining are contributions from subsets in $\mathcal{S}_0$: a subset of size i spans a forest with $n - i$ components, thus contributing $(-1)^i z^{n-i}$ to the sum. $\square$

Proposition 2.25. *Suppose G is a simple connected graph on n vertices and m edges and having girth g, and that $P(G;z) = \sum(-1)^i b_i(G) z^{n-i}$. Then*

$$b_i(G) = \binom{m}{i}, \text{ for } i = 0, 1, \ldots, g-2,$$

and

$$b_{g-1}(G) = \binom{m}{g-1} - t,$$

where t is the number of g-cycles of G.

Proposition 2.26. *If G is a simple connected graph on n vertices and m edges and $P(G;z) = \sum(-1)^i b_i(G) z^{n-i}$ then, for $0 \le i \le n-1$,*

$$\binom{n-1}{i} \le b_i(G) \le \binom{m}{i}.$$

Proof. For the lower bound choose a spanning tree of G and label its edges with $1, \ldots, n-1$, the remaining edges with larger labels $n, \ldots, m$. Any edge subset of T contains no broken circuit. □

Proposition 2.27. *If G is a simple connected graph on n vertices and m edges and $P(G;z) = \sum(-1)^i b_i(G) z^{n-i}$ then,*

$$b_{i-1}(G) \le b_i(G) \ \textit{ for all } \ 1 \le i \le \frac{1}{2}(n-1).$$

Proof. In terms of the coefficients relative to the tree basis $\{z(z-1)^{n-1}\}$,

$$P(G;z) = \sum_{i=0}^{n-1} (-1)^{n-1-i} t_{i,0}(G) z(z-1)^i,$$

we have

$$b_i(G) = \sum_{j=0}^{i} t_{n-1-j,0}(G) \binom{n-1-j}{n-1-i} = \sum_{j=0}^{i} t_{n-1-j,0}(G) \binom{n-1-j}{i-j}.$$

If $i \le \frac{1}{2}(n-1)$ then $i - j \le \frac{1}{2}(n-1-j)$ for all $j \ge 0$. By unimodality of the binomial coefficients,

$$\binom{n-1-j}{i-j} \ge \binom{n-1-j}{i-1-j} \text{ for } \ i \le \frac{1}{2}(n-1), \ j \ge 0.$$

Since each $t_{n-1-j,0}(G)$ is a non-negative integer, it follows that $b_i(G) \ge b_{i-1}(G)$ for $i \le \frac{1}{2}(n-1)$. □

Proposition 2.27 is the easy half of a long-standing conjecture first made by Read in 1968 that the coefficients $b_i(G)$ of the chromatic polynomial are *unimodal*. An even stronger conjecture of *log-concavity* was later made, i.e., that $b_{i-1}(G) b_{i+1}(G) \le b_i(G)^2$. This has recently been proved by J. Huh [28].

Exercise 2.28. *Prove that for a connected graph G on n vertices and with m edges,*

$$b_i(G) \ge \binom{n-1}{i} + \binom{n-2}{i-1}(m-n+1)$$

for $0 \le i \le n-1$.

[Hint: use the expression for $b_i(G)$ in terms of the $t_{i,0}(G)$ that was used in the proof of the last proposition.]

2.4 Some other deletion–contraction invariants.

We have seen that the chromatic polynomial $P(G;z)$ satisfies the recurrence relation

$$P(G;z) = P(G\backslash e) - P(G/e;z), \tag{2}$$

for any edge e of G. Together with boundary conditions

$$P(\overline{K}_n;z) = z^n, \qquad n = 1,2,\dots \tag{3}$$

this suffices to determine $P(G;z)$ on all graphs. A slight variation on giving the boundary conditions (3) is to supplement the recurrence (2) with the natural property of multiplicativity over disjoint unions

$$P(G_1 \cup G_2;z) = P(G_1;z)P(G_2;z), \tag{4}$$

and then to give the single boundary condition $P(K_1;z) = z$.

Define

$$B(G;k,y) = \sum_{f:V(G)\to[k]} y^{\#\{uv\in E(G):f(u)=f(v)\}},$$

where $k \in \mathbb{Z}_{>0}$ and y is an indeterminate. This polynomial in y is a generating function for colourings of G (not necessarily proper) counted according to the number of monochromatic edges, i.e., edges receiving the same colour on their endpoints. (Edges are taken with their multiplicity when counting the number of monochromatic edges in the exponent of y.) Note that $B(G;k,0) = P(G;k)$.

Proposition 2.29. *For each edge e of G,*

$$B(G;k,y) = (y-1)B(G/e;k,y) + B(G\backslash e;k,y).$$

Together with the boundary conditions $B(\overline{K}_n;k,y) = k^n$, for $n = 1,2,\dots,$ this determines $B(G;k,y)$ as a polynomial in k and y.

Proof. Given $e = st$,

$$\begin{aligned} B(G;k,y) &= y \sum_{\substack{f:V(G)\to[k]\\ f(s)=f(t)}} y^{\#\{uv\in E\backslash e:f(u)=f(v)\}} + \sum_{\substack{f:V(G)\to[k]\\ f(s)\neq f(t)}} y^{\#\{uv\in E\backslash e:f(u)=f(v)\}} \\ &= yB(G/e;k,y) + [B(G\backslash e;k,y) - B(G/e;k,y)]. \end{aligned}$$

The fact that $B(G;k,y)$ is a polynomial follows by induction of the number of edges and the given boundary condition $B(\overline{K}_n;k,y) = k^n$. Further, it has degree $|V(G)|$ as a polynomial in k and degree $|E(G)|$ as a polynomial in y (again by induction on number of edges by tracking the relevant coefficient in the recurrence $B(G;k,y) = (y-1)B(G/e;k,y) + B(G\backslash e;k,y)$). □

An *acyclic orientation* of a graph is an orientation that has no directed cycles. A loop has no acyclic orientation, but any loopless graph does (for example, if its vertices are labelled by $1,\dots,n$ and an edge is directed from the smaller to the higher number).

Theorem 2.30. [Stanley, 1973] *The number of acyclic orientations of a graph G with at least one edge is given by $(-1)^{|V(G)|}P(G;-1)$.*

Proof. Let $Q(G)$ denote the number of acyclic orientations of G. When G is a single edge $Q(G) = 2$ and when G is a loop $Q(G) = 0$. If e is parallel to another edge of G then $Q(G) = Q(G\backslash e)$, since parallel edges must have the same direction in an acyclic orientation. Also, Q is multiplicative over disjoint unions, i.e., $Q(G_1 \cup G_2) = Q(G_1)Q(G_2)$.

To prove then that $Q(G)) = (-1)^{|V(G)|}P(G;-1)$ it suffices to show that when e is not a loop or parallel to another edge of G we have

$$Q(G) = Q(G\backslash e) + Q(G/e). \tag{5}$$

Let $e = uv$ be a simple edge of G and consider an acyclic orientation $\mathcal{O}$ of $G\backslash e$. There is always one direction $u \to v$ or $u \leftarrow v$ possible so that $\mathcal{O}$ can be extended to an acyclic orientation of G: if both directions were to produce directed cycles then there would have to be a directed path from u to v and a directed path from v to u, which together would make a directed cycle in $\mathcal{O}$.

Those acyclic orientations of $G\backslash e$ that permit exactly one direction of e are in bijective correspondence with the subset of acyclic orientations of G where the direction of e cannot be reversed while preserving the property of being acyclic. Such an orientation of G induces an orientation that has a directed cycle in G/e, and contributes 1 to $Q(G)$ and $1 + 0 = 1$ to $Q(G\backslash e) + Q(G/e)$.

Those acyclic orientations of $G\backslash e$ where the direction of e can be reversed to make another acyclic orientation of G are in bijective correspondence with those orientations of G that induce acyclic orientations on the contracted graph G/e. Such a pair of acyclic orientations of G differing just on the direction of e contribute 2 to $Q(G)$ and $1 + 1 = 2$ to $Q(G\backslash e) + Q(G/e)$.

This establishes the recurrence (5). □

In [47] Tutte describes how he was led to define his polynomial (he called it the dichromate) by observing how graph invariants such as the chromatic polynomial and the number of spanning trees of a graph shared the property of satisfying a deletion–contraction recurrence.

Exercise 2.31. *Suppose $f(G)$ is a graph invariant that for a connected graph G counts one of the following:*

(i) the number of spanning trees of G,

(ii) the number of spanning forests of G,

(iii) the number of connected spanning subgraphs of G.

Further suppose we stipulate that f is multiplicative over disjoint unions, $f(G_1 \cup G_2) = f(G_1)f(G_2)$.

Show that in each case f satisfies the recurrence

$$f(G) = f(G\backslash e) + f(G/e),$$

for each edge e of G that is not a loop or bridge. How do these three invariants differ for bridges and loops?

3 The Tutte polynomial.

Recall that the rank of G is defined by $r(G) = |V(G)| - c(G)$. The *nullity* of G is defined by $n(G) = |E(G)| - r(G)$.

It will be convenient to call an edge *ordinary* when it is neither a bridge nor a loop.

Consider the following recursive definition of a graph invariant $T(G;x,y)$ in two independent variables x and y. If G has no edges then $T(G;x,y)=1$, otherwise, for any $e \in E(G)$,

$$T(G;x,y) = \begin{cases} T(G/e;x,y) + T(G\backslash e;x,y) & e \text{ ordinary,} \\ xT(G/e;x,y) & e \text{ a bridge,} \\ yT(G\backslash e;x,y) & e \text{ a loop.} \end{cases} \tag{6}$$

By induction this defines a bivariate polynomial $T(G;x,y)$, called the *Tutte polynomial* of G, all of whose coefficients are non-negative integers.

It is not immediately clear that it does not matter which order the edges are chosen to calculate $T(G;x,y)$ recursively using (6).

Proposition 3.1. *If e_1 and e_2 are distinct edges of G then the outcome of first applying the recurrence* (6) *with edge e_1 and then with edge e_2 is the same as with the reverse order, when first taking e_2 and then e_1.*

Proof. (Sketch) First observe that if e_1 and e_2 are parallel then the statement is clearly true (swapping e_1 and e_2 is an automorphism of G). When e_1 and e_2 are not parallel, the type of edge e_2 in G (whether it is a bridge, loop, or ordinary) is preserved in G/e_1 and in $G\backslash e_1$. For each of the possible combinations of edge types for e_1 and e_2, one verifies that swapping the order of e_1 and e_2 gives the same outcome in the two-level computation tree going from G to G with edges e_1 and e_2 deleted or contracted. For example, if both edges are ordinary then the truth of the statement amounts to the fact that $G/e_1\backslash e_2 \cong G\backslash e_2/e_1$ and similarly for the other three combinations of deletion and contraction. $\square$

The recurrence (6) can be restated as follows. If G consists of k bridges and ℓ loops then $T(G;x,y)=x^k y^\ell$, otherwise $T(G;x,y)=T(G/e;x,y)+T(G\backslash e;x,y)$ for an ordinary edge e of G.

A graph is 2-connected if it has no cut-vertex. Note that K_2 is 2-connected, and so is a single vertex with a loop. A graph with a loop or bridge and at least one other edge is not 2-connected. If G is 2-connected and H is a minor of G then $t_{i,j}(H) \le t_{i,j}(G)$ (proved by Brylawski [8, Corollary 6.9] in the more general context of matroids). When G is not 2-connected and H is a minor of G, the coefficients of $T(H;x,y)$ are not necessarily dominated by those of $T(G;x,y)$: for an example, take G to be a tree and H any proper minor.

A *block* of G is a maximal 2-connected induced subgraph of G. A bridge or a loop of G is a block of G. If G is not 2-connected then it can be written in the form $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| \le 1$, i.e., where $(V(G_1), V(G_2))$ is a 1-separation of G. The intersection graph of the blocks of a connected loopless graph is a tree. In particular, if G is connected and has at least two blocks then there are at least two *endblocks* of G which are blocks containing only one cut-vertex of G.

Proposition 3.2. *The Tutte polynomial of G is multiplicative over the connected components of G and over the blocks of G: if $G = G_1 \cup G_2$ where G_1 and G_2 share at most one vertex then $T(G_1 \cup G_2;x,y) = T(G_1;x,y)T(G_2;x,y)$.*

Proof. The statement is true when each edge is either a bridge or a loop, since in this case $T(G;x,y)=x^k y^\ell$, where k is the number of bridges and ℓ the number of loops. We argue by induction on the number of ordinary edges of G. Let $G = G_1 \cup G_2$ where $(V(G_1), V(G_2))$ is a 1-separation of G. The endpoints of any edge e must belong to the same block of G; if e is a bridge then it forms its own block. Suppose $G = G_1 \cup G_2$ where G_1 is a block of G containing an ordinary edge e. Deleting or contracting e can only decrease the number of ordinary edges of G and since e is ordinary we have, writing $T(G;x,y)=T(G)$,

$$\begin{aligned} T(G) &= T(G/e) + T(G\backslash e) \\ &= T(G_1/e \cup G_2) + T(G_1\backslash e \cup G_2) \\ &= [T(G_1/e) + T(G_1\backslash e)]T(G_2) \\ &= T(G_1)T(G_2), \end{aligned}$$

where to obtain the third line we applied the induction hypothesis. □

The converse to Proposition 3.2 also holds, although its proof is bit more involved:

Theorem 3.3. [37] *If G is 2-connected graph without loops then $T(G;x,y)$ is irreducible in $\mathbb{Z}[x,y]$.*

The factors of the Tutte polynomial of G therefore correspond precisely to the blocks of G and any loops (each contributing a factor y).

As well as being multiplicative over blocks and connected components, and so unaffected by the operation of identifying vertices in different connected components of G, the Tutte polynomial is also unaffected by Whitney twists:

Proposition 3.4. *If G and G' are 2-isomorphic then $T(G;x,y)=T(G';x,y)$. (The Tutte polynomial of G only depends on the cycle matroid of G.)*

Here are some basic properties of the coefficients of $T(G;x,y)$:

Proposition 3.5. *For a graph G with Tutte polynomial $T(G;x,y)=\sum t_{i,j}(G)x^iy^j$,*

(i) $t_{0,0}(G)=0$ *if* $|E(G)|>0$;

(ii) if G has no loops then $t_{1,0}(G)\neq 0$ if and only if G is 2-connected;

(iii) x^k divides $T(G;x,y)$ if and only if G has at least k bridges, and y^ℓ divides $T(G;x,y)$ if and only if G has at least ℓ loops;

(iv) given G has k bridges and ℓ loops, if $i\geq r(G)$ or $j\geq n(G)$ then $t_{i,j}(G)=0$ except when $i=r(G)$ and $j=\ell$, or $i=k$ and $j=n(G)$, where we have $t_{r(G),\ell}(G)=1=t_{k,n(G)}(G)$.

Proof. For (ii), we use the property that if G is 2-connected, then at least one of G/e and $G\backslash e$ is also 2-connected. A basis for induction is that $T(K_2;x,y)=x$. Given a loopless graph G, if e is not parallel to another edge then both G/e and $G\backslash e$ have no loops, and the equation $t_{1,0}(G)=t_{1,0}(G/e)+t_{1,0}(G\backslash e)$ provides the inductive step. If e is parallel to another edge then G/e has a loop and $t_{1,0}(G)=t_{1,0}(G\backslash e)$; by deleting all but one edge in a parallel class we can thus assume G is simple. For the converse, if G is not 2-connected then by Proposition 3.2 its Tutte polynomial is the product of at least two polynomial factors, each corresponding to a block of G; by what we have just proved $t_{1,0}(B)=1$ for each such block B, and this implies $t_{1,0}(G)=0$.

For (iv), we shall use induction on the number of ordinary edges to prove that $t_{i,j}(G)=0$ when $i\geq r(G)$ or $j\geq n(G)$, except for $t_{r(G),\ell}(G)=1=t_{k,n(G)}(G)$. The base case is when G has no ordinary edges, consisting of k bridges and ℓ loops. Here $r(G)=k$ and $n(G)=\ell$, and $t_{k,\ell}(G)=1$, while $t_{i,j}(G)=0$ for all other values of i,j. Hence the statement is true in this case.

Consider the recurrence formula $t_{i,j}(G)=t_{i,j}(G/e)+t_{i,j}(G\backslash e)$ for ordinary edge e. We have by inductive hypothesis that $t_{i,j}(G/e)=0$ for $i\geq$

$r(G/e) = r(G)-1$ except $t_{r(G)-1,\ell}(G/e) = 1$, and for $j \geq n(G/e) = n(G)$ except $t_{k,n(G)}(G/e) = 1$. This gives $t_{i,j}(G) = 0$ for $j \geq n(G)$ except $t_{k,n(G)}(G) = 1$.

Also $t_{i,j}(G\backslash e) = 0$ for $i \geq r(G\backslash e) = r(G)$ except $t_{r(G),\ell}(G\backslash e) = 1$, and for $j \geq n(G\backslash e) = n(G) - 1$ except $t_{k,n(G)-1}(G/e) = 1$. This gives $t_{r(G),\ell}(G) = 0$ for $i \geq r(G)$ except $t_{r(G),\ell}(G) = 1$. $\square$

3.1 Evaluations of the Tutte polynomial.

We have seen in Theorem 2.30 that the number of acyclic orientations is the evaluation of the Tutte polynomial at $x = 2$ and $y = 0$. Also, by Exercise 2.31 the number of spanning trees, spanning forests and connected spanning subgraphs are each equal to an evaluation of the Tutte polynomial. The following "Recipe Theorem" describes the necessary ingredients for other evaluations of the Tutte polynomial.

Theorem 3.6. *Let $\mathcal{G}$ be a minor-closed class of graphs. There is a unique graph invariant $f : \mathcal{G} \to \mathbb{Z}[x, y, \alpha, \beta, \gamma]$ such that $f(\overline{K}_n) = \gamma^n$ for $n = 1, 2, \ldots$, and for every edge $e \in E$*

$$f(G) = \begin{cases} \alpha f(G/e) + \beta f(G\backslash e) & e \text{ not a bridge or loop,} \\ x f(G/e) & e \text{ a bridge,} \\ y f(G\backslash e) & e \text{ a loop.} \end{cases} \tag{7}$$

The graph invariant f is equal to the following specialization of the Tutte polynomial:

$$f(G) = \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \tag{8}$$

Note. (i) If instead of contracting a bridge we require that $f(G) = xf(G\backslash e)$ when e is a bridge, the Tutte polynomial is evaluated at the point $(\gamma x/\alpha, y/\beta)$ instead of $(x/\alpha, y/\beta)$. In particular, when $\gamma = 1$ it does not matter whether bridges are deleted or contracted.

(ii) If either α or β is zero then we interpret (8) as the result of substituting values of the parameters after expanding the expression on the right-hand side as a polynomial in $\mathbb{Z}[\alpha, \beta, \gamma, x, y]$. Given a graph G with k bridges and ℓ loops, using Proposition 3.5 (iv) we see that if $\alpha = 0$ then $f(G) = \gamma^{c(G)} \beta^{n(G)-\ell} x^{r(G)} y^{\ell}$, and if $\beta = 0$ then $f(G) = \gamma^{c(G)} \alpha^{r(G)-k} x^k y^{n(G)}$. If both α and β are zero then $f(G) = 0$ if G has an ordinary edge, while $f(G) = \gamma^{c(G)} x^k y^{\ell}$ if $E(G)$ consists of just k bridges and ℓ loops.

Proof. Uniqueness of $f(G)$ follows by induction on the number of edges and application of the recurrence (7).

Formula (8) is certainly true for cocliques $\overline{K}_n$. If G consists just of k bridges and ℓ loops and has c connected components, then $f(G) = \gamma^c x^k y^{\ell}$ and since $r(G) = k$ and $n(G) = \ell$ we have $T(G; \frac{x}{\alpha}, \frac{y}{\beta}) = \left(\frac{x}{\alpha}\right)^k \left(\frac{y}{\beta}\right)^{\ell}$, so (8) is satisfied. Let e be an ordinary edge, and note that $c(G) = c(G/e) = c(G\backslash e)$, so that $r(G/e) = r(G) - 1$, $r(G \setminus e) = r(G)$ and $n(G/e) = n(G)$, $n(G\backslash e) = n(G) - 1$. By induction on the number of ordinary edges,

$$\begin{aligned} f(G) &= \alpha f(G/e) + \beta f(G\backslash e) \\ &= \alpha \cdot \gamma^{c(G)} \alpha^{r(G)-1} \beta^{n(G)} T(G/e; \frac{x}{\alpha}, \frac{y}{\beta}) + \beta \cdot \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)-1} T(G\backslash e; \frac{x}{\alpha}, \frac{y}{\beta}) \\ &= \gamma^{c(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}). \end{aligned}$$

$\square$

A graph invariant satisfying the recurrence (7) is called a *generalized Tutte–Grothendieck invariant*, or TG-invariant for short. A TG-invariant is multiplicative over disjoint unions, and if G_1 and G_2 share just one vertex then $f(G_1 \cup G_2) = f(G_1)f(G_2)/\gamma$. (The archetypal example is the chromatic polynomial.) See [9] for TG-invariants in graph theory and matroid theory more generally.

Exercise 3.7. *Suppose the graph invariant $f(G)$ satisfies the recurrence* (7)*. Show that the graph invariant*

$$\left(\frac{x-\alpha}{\beta\gamma}\right)^{c(G)}\left(\frac{y-\beta}{\alpha}\right)^{|V(G)|}\delta^{|E(G)|}f(G),$$

where δ is an arbitrary constant, satisfies the recurrence $f(G) = (y-\beta)f(G/e) + \beta f(G\backslash e)$ independently of whether e is a bridge, loop, or ordinary. (For example, the chromatic polynomial is an example of such an invariant, its recurrence $P(G;z) = P(G\backslash e;z) - P(G/e;z)$ holding for any edge e.)

An example we have already seen for Exercise 3.7 is when $f(G)$ is the number of acyclic orientations of G. This is a TG-invariant with $\alpha = \beta = \gamma = 1$ and $x = 2, y = 0$, satisfying $f(G) = f(G/e) + f(G\backslash e)$ when e is not a loop. The invariant $(-1)^{|V(G)|}f(G)$ satisfies $f(G) = f(G \setminus e) - f(G/e)$ for all edges e (as we know from Theorem 2.30, it is equal to $P(G;-1)$).

Proposition 3.8. *The* monochrome polynomial*,*

$$B(G;k,y) = \sum_{f:V(G)\to[k]} y^{\#\{uv\in E(G):f(u)=f(v)\}},$$

is the following specialization of the Tutte polynomial:

$$B(G;k,y) = k^{c(G)}(y-1)^{r(G)}T(G;\frac{y-1+k}{y-1},y).$$

The chromatic polynomial is given by

$$P(G;z) = (-1)^{r(G)}z^{c(G)}T(G;1-z,0).$$

Proof. Proposition 2.29 gives the recurrence formula

$$B(G;k,y) = (y-1)B(G/e;k,y) + B(G\backslash e;k,y), \tag{9}$$

valid for all edges e.

For the chromatic polynomial we have $P(G;z) = (z-1)P(G/e;z)$ when e is a bridge, for by Proposition 2.22 we have $P(G\backslash e;z) = zP(G/e;z)$. A direct argument for $P(G \setminus e;k) = kP(G/e;k)$ when $e = uv$ is a bridge is as follows. Suppose $G\backslash e = G_1 \cup G_2$ with $u \in V(G_1)$ and $v \in V(G_2)$. Then G/e is obtained from $G_1 \cup G_2$ by identifying the vertices u and v to make a cut-vertex w. Given a fixed colour $\ell \in [k]$, there are $P(G_1;k)/k$ proper colourings $f_1 : V(G_1) \to [k]$ of G_1 with $f_1(w) = \ell$, and $P(G_2;k)/k$ proper colourings $f_2 : V(G_2) \to [k]$ of G_2 with $f_2(w) = \ell$. Since there are no edges between G_1 and G_2, there are $P(G_1;k)P(G_2;k)/k^2$ proper colourings of G/e with $f(w) = \ell$. This number is independent of ℓ, so there are $P(G_1;k)P(G_2;k)/k$ proper colourings of G/e. On the other hand, there are $P(G_1;k)P(G_2;k)$ proper colourings of $G\backslash e$. Hence $P(G\backslash e;k) = kP(G/e;k)$ when e is a bridge of G.

For the monochrome polynomial, when e is a bridge we have $B(G\backslash e; k, y) = kB(G/e; k, y)$, by a similar argument to the chromatic polynomial, by conditioning on the colour of the cut-vertex w of G/e obtained by identifying the endpoints of e. Instead of proper colourings, consider colourings with exactly m_1 monochrome edges in G_1 and exactly m_2 monochrome edges in G_2. Then the number of such colourings for $G\backslash e$ (the disjoint union of G_1 and G_2) is k times the number for G/e (the gluing of G_1 and G_2 at a vertex). Collecting together all colourings for which $m_1 + m_2 = m$, this implies that the coefficient of y^m in $B(G\backslash e; k, y)$ is equal to k times the corresponding coefficient in $B(G/e; k, y)$. Since this holds for each m, it follows that $B(G\backslash e; k, y) = kB(G/e; k, y)$ when e is a bridge, and so $B(G; k, y) = (y-1+k)B(G/e)$ by the recurrence formula (9). When e is a loop $B(G; k, y) = yB(G\backslash e; k, y)$ since a loop is always monochromatic (or by looking at the recurrence formula (9) with $G/e \cong G\backslash e$ when e is a loop).

The result now follows by Proposition 3.6. $\square$

We have seen that $T(G; 2, 0) = (-1)^{|V(G)|}P(G; -1)$ counts acyclic orientations of G. An acyclic orientation of G has at least one source (all edges outgoing) and at least one sink (all edges incoming).

Theorem 3.9. [Greene and Zavslasky, 1983] *Suppose G is a connected graph and $u \in V(G)$. Then the number of acyclic orientations of G with unique source at u is equal to $T(G; 1, 0)$. In particular, this number is independent of the choice of u.*

Note that $T(G; 1, 0) = P'(G; 0)$, the coefficient of z in $P(G; z)$, when G is connected.

Proof. Fix a vertex u of G and let $Q_u(G)$ denote the number of acyclic orientations with a unique source at u.

Suppose G is connected and with at least one edge. Choose an edge $e = uv$ with one endpoint the source vertex u. (Since G is connected there has to be at least one edge incident with u.)

If e is the only edge of G, then $Q_u(G) = 1$ when e is a bridge, and $Q_u(G) = 0$ when e is a loop. Suppose there are other edges.

If e is a loop then $Q_u(G) = 0$.

If e is a bridge then $Q_u(G) = Q_u(G/e)$. For consider an acyclic orientation $\mathcal{O}$ of G with unique source u. Then in the component of $G\backslash e$ containing v, the only source of $\mathcal{O}$ restricted to this component has to be v, otherwise there would be a source other than u in $\mathcal{O}$. Therefore, acyclic orientations of G with unique source at u are in one-to-one correspondence with acyclic orientations of G/e with unique source at u (which in G/e has been identified with the vertex v).

If e is ordinary then partition acyclic orientations with u as a unique source into two sets: those for which uv is the only edge directed into v (so deleting uv does not give an acyclic orientation of $G\backslash e$ with a unique source) and those for which uv is not the only edge directed into v (here deleting uv gives an acyclic orientation of $G\backslash e$ with unique source at u). The first set is in one-to-one correspondence with acyclic orientations of G/e with unique source at u (in G/e vertex v is identified with vertex u), while the second set is in one-to-one correspondence with acyclic orientations of $G\backslash e$ with unique source at u. Hence when e is ordinary we have $Q_u(G) = Q_u(G/e) + Q_u(G\backslash e)$.

By Proposition 3.6 it follows that $Q_u(G) = T(G; 1, 0)$. $\square$

Consider a connected graph $G = (V, E)$ in which each edge is deleted independently at random with probability $1-p$ (e remains with probability p). The probability that G remains connected is known as the *(all-terminal) reliability* $R(G;p)$ and is given by

$$R(G;p) = \sum_A p^{|A|}(1-p)^{|E\backslash A|},$$

where the sum is over all spanning connected subgraphs (V, A).

Proposition 3.10. *If $G = (V, E)$ is a connected graph then*

$$R(G;p) = (1-p)^{|E|-|V|+1}p^{|V|-1}T(G;1,\frac{1}{1-p}).$$

Proof. Establish the recurrence

$$R(G;p) = pR(G/e;p) + (1-p)R(G\backslash e),$$

by conditioning on the events that e is or is not deleted. By Theorem 3.6 the result follows. $\square$

When G is not connected the appropriate event to consider is whether G still has the same number of connected components after independently deleting edges at random with probability $1-p$, i.e., whether its rank of G is preserved. The probability of this event is $(1-p)^{n(G)}p^{r(G)}T(G;1,\frac{1}{1-p})$, by multiplicativity of this invariant over connected components.

3.2 Subgraph expansion.

Let $G = (V, E)$ be a graph and $A \subseteq E$. Identify A with the spanning subgraph $G_A = (V, A)$. The rank of A is defined by $r_G(A) = |V(G)| - c(G_A)$. Thus $r_G(E) = r(G)$ in the notation already introduced for the rank of the graph G. When context makes it clear what graph G is, we drop the subscript and write $r(A)$ for $r_G(A)$.

It is easy to see that $0 \leq r(A) \leq |A|$ with $r(A) = 0$ if and only if $A = \emptyset$ and $r(A) = |A|$ if and only if G_A is a forest. Also, $A \subseteq B$ implies $r(A) \leq r(B)$ and $r(A) = r(E)$ if and only if $c(G_A) = c(G)$.

Proposition 3.11. *The Tutte polynomial of a graph $G = (V, E)$ has subgraph expansion*

$$T(G;x,y) = \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{10}$$

Proof. Set

$$R(G;u,v) = \sum_{A\subseteq E}u^{r(E)-r(A)}v^{|A|-r(A)},$$

(the Whitney rank-nullity generating function for G). We wish to prove that $T(G;x,y) = R(G;x-1,y-1)$ and shall do this by verifying that $R(G;u,v)$ satisfies the TG-invariant recurrence formula: (i) $R(G;u,v) = 1$ if $E = \emptyset$, (ii) $R(G;u,v) = (u+1)R(G\backslash e;u,v)$ when e is a bridge, (iii) $R(G;u,v) = (v+1)R(G\backslash e;u,v)$ when e is a loop, and (iv) $R(G;u,v) = R(G/e;u,v)+R(G\backslash e;u,v)$ when e is ordinary.

When $E = \emptyset$ we have $R(G;u,v) = 1$.

If $e \notin A$ then

$$r_G(A) = r_{G\backslash e}(A). \tag{11}$$

If $e \in A$ then

$$r_{G\backslash e}(A\backslash e) = \begin{cases} r_G(A) - 1 & \text{if } e \text{ is a bridge,} \\ r_G(A) & \text{if } e \text{ is a loop,} \end{cases} \tag{12}$$

and

$$r_{G/e}(A\backslash e) = r_G(A) - 1 \quad \text{ if } e \text{ is ordinary or a bridge.} \tag{13}$$

Suppose e is a bridge. Then by (11) and (12),

$$\begin{aligned} R(G;u,v) &= \sum_{A\subseteq E\backslash e} u^{r_G(E)-r_G(A)}v^{|A|-r_G(A)} + \sum_{e\in A\subseteq E} u^{r_G(E)-r(A)}v^{|A|-r_G(A)} \\ &= u \sum_{A\subseteq E\backslash e} u^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)}v^{|A|-r_{G\backslash e}(A)} \\ &\qquad + \sum_{B=A\backslash e} u^{r_{G\backslash e}(E\backslash e)+1-(r_{G\backslash e}(B)+1)}v^{|B|+1-(r_{G\backslash e}(B)+1)} \\ &= (u+1)R(G\backslash e;u,v). \end{aligned}$$

The case when e is a loop is similarly argued.

When e is ordinary, by (11) and (13),

$$\begin{aligned} R(G;u,v) &= \sum_{A\subseteq E\backslash e} u^{r_G(E)-r_G(A)}v^{|A|-r_G(A)} + \sum_{e\in A\subseteq E} u^{r_G(E)-r_G(A)}v^{|A|-r_G(A)} \\ &= \sum_{A\subseteq E\backslash e} u^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)}v^{|A|-r_{G\backslash e}(A)} \\ &\qquad + \sum_{B=A\backslash e} u^{r_{G/e}(E\backslash e)+1-(r_{G/e}(B)+1)}v^{|B|+1-(r_{G/e}(B)+1)} \\ &= R(G\backslash e;u,v) + R(G/e;u,v). \end{aligned}$$

□

It is common to *define* the Tutte polynomial by its subgraph expansion (10), having over the deletion–contraction formulation (6) the advantage of being unambiguously well-defined. On the other hand, it is not apparent from (10) that the coefficients of the Tutte polynomial are non-negative integers, and often it is easier to derive a combinatorial interpretation for an evaluation of the Tutte polynomial by using the deletion–contraction recurrence. Nonetheless, it is easy to read off the following evaluations of the Tutte polynomial from its subgraph expansion.

Proposition 3.12. *Let G be a connected graph. Then*

$$\begin{aligned} T(G;1,1) &= \#\text{spanning trees}, \\ T(G;2,1) &= \#\text{spanning forests}, \\ T(G;1,2) &= \#\text{connected spanning subgraphs}, \\ T(G;2,2) &= 2^{|E|}. \end{aligned}$$

If $(x-1)(y-1) = 1$ then $T(G;x,y) = (x-1)^{r(E)}y^{|E|}$.

Along the hyperbola $(x-1)(y-1) = z$ we have, for graph $G = (V,E)$,

$$\begin{aligned} T(G;x,y) &= (y-1)^{-|V|} \sum_{A\subseteq E} \left(\frac{z}{y-1}\right)^{c(G_A)-c(G)} (y-1)^{|A|+c(G_A)} \\ &= (y-1)^{-r(G)} z^{-c(G)} \sum_{A\subseteq E} z^{c(G_A)}(y-1)^{|A|}. \end{aligned}$$

When $y = 0$ this is the subgraph expansion for the chromatic polynomial that we obtained ealier by inclusion–exclusion. The polynomial $\sum_{A \subseteq E} z^{c(G_A)} w^{|A|}$ is the partition function for the Fortuin–Kasteleyn random cluster model in statistical physics (the normalizing constant for a probability space on subgraphs of G, the probability of $G_A = (V, A)$ depending on both $|A|$ and $c(A)$). This model generalizes the k-state Potts model, which is the case $z = k \in \mathbb{Z}_+$, and whose partition function we have already met in the form of the monochrome polynomial $B(G; k, y)$.

For a connected graph $G = (V, E)$,

$$xT(G; x+1, 1) = \sum_{\substack{A \subseteq E \\ n(A)=0}} x^{c(G_A)}$$

is the generating function for spanning forests of G by number of connected components, and

$$y^{|V|-1} T(G; 1, y+1) = \sum_{\substack{A \subseteq E \\ c(G_A)=c(G)}} y^{|A|}$$

is the generating function for connected spanning subgraphs of G by size.

3.3 Coefficients. Spanning tree expansion.

A graph invariant is called a *Tutte invariant* if it can be found as some function of the coefficients of $T(G; x, y)$. Thus the property of having at least one edge is a Tutte invariant since $t_{0,0}(G) = 0$ if and only if G has an edge. In fact $|E|$ is itself a Tutte invariant since $r(G) = \max\{i : t_{i,j}(G) \neq 0\}$ and $n(G) = \max\{j : t_{i,j}(G) \neq 0\}$ are Tutte invariants and $r(G) + n(G) = |E|$. For another example, from Proposition 3.5 (ii), a loopless graph G is 2-connected if and only if $t_{1,0}(G) \neq 0$.

Examples of graph invariants that are not Tutte invariants include the degree sequence of G and whether G is planar. A tree on n vertices has Tutte polynomial x^{n-1}, and for $n \geq 3$ there are two trees on n vertices with different degree sequences. Less trivially, there are non-2-isomorphic graphs G and G' which have different degree sequences. Likewise, there is a planar graph G and non-planar graph G' with $T(G; x, y) = T(G'; x, y)$. (See [36, Appendix] for examples.)

In this section we shall give Tutte's 1954 inductive proof that, for a connected graph G, the coeffficients $t_{i,j}(G)$ count a certain subset of the spanning trees of G. The interpretation of $t_{i,j}(G)$ when G is not necessarily connected follows as an easy consequence of multiplicativity of $T(G; x, y)$ over disjoint unions. A subgraph $G_A = (V, A)$ has $r(A) = r(E)$ and $n(A) = 0$ if and only if G_A is a *maximal spanning forest*, in the sense that no edge can be added to G_A without creating a cycle, i.e., G_A consists of a spanning tree of each connected component of G.

Let $G = (V, E)$ be a connected graph and T a spanning tree of G. Then

(i) for each $e \in E \setminus T$ there is a unique cycle in G contained in $T \cup \{e\}$, which we shall denote by $\mathrm{cyc}(T, e)$, and

(ii) for each $e \in T$ there is a unique cut contained in $E \setminus T \cup \{e\}$, which we shall denote by $\mathrm{cut}(T, e)$.

Put a linear order $<$ on E. Say $E = \{e_1, e_2, \ldots, e_m\}$, where $e_1 < e_2 < \cdots < e_m$.

Definition 3.13. *Given a spanning tree T of a connected graph G with an ordering of its edges, an edge $e \in T$ is* internally active *with respect to T if e is the least edge in* $\mathrm{cut}(T, e)$. *An edge $e \in E \setminus T$ is* externally active *with respect to T if e is the least edge in* $\mathrm{cyc}(T, e)$. *A spanning tree T has* internal activity *i and* external activity *j when there are precisely i internally active edges with respect to T and j externally active edges with respect to T.*

Tutte was led to his spanning tree expansion of the Tutte polynomial of a connected graph by observing that in the recursive definition of $T(G; x, y)$, if one applies deletion and contraction to edges of E in reverse order $e_m, e_{m-1}, \ldots, e_2, e_1$, the result will be an expression for $T(G; x, y)$ as a sum in which each summand is obtained by contracting the elements in some spanning tree T of G and deleting the elements of $E \setminus T$. Moreover, in the process of obtaining this summand the edges contracted as bridges will be precisely the internally active edges with respect to T, and the elements of E deleted as loops will be precisely the externally active edges with respect to T.

Theorem 3.14. [Tutte, 1954] *Let G be a connected graph with an order on its edges and for each $0 \leq i \leq |V| - 1, 0 \leq j \leq |E| - |V| + 1$ let $t_{i,j}(G)$ denote the number of spanning trees of G of internal activity i and external activity j. Then the Tutte polynomial of G is equal to*

$$T(G; x, y) = \sum t_{i,j}(G) x^i y^j.$$

In particular, $t_{i,j}(G)$ is a graph invariant, independent of the ordering of the edges of G.

Proof. We proceed by induction on the number of edges of G.

When there are no edges in G, i.e., $G \cong K_1$, we have $t_{0,0}(G) = 1$ and $t_{i,j}(G) = 0$ for $i + j > 0$.

Let $G = (V, E)$, $E = \{e_1 < e_2 < \ldots < e_m\}$, $m \geq 1$, and assume the assertion holds for connected graphs with at most $m - 1$ edges.

The graphs G/e_m and $G \backslash e_m$ are both connected when e_m is ordinary or a loop, while only G/e_m is connected when e_m is a bridge, but this is fine because we only contract bridges in the recurrence for $T(G; x, y)$. We take $E(G/e_m) = E(G \backslash e_m) = \{e_1 < e_2 < \cdots < e_{m-1}\}$.

(i) Suppose e_m is a bridge. Then e_m is in every spanning tree of G, and a subgraph T is a spanning tree if and only if $e_m \in T$ and T/e_m is a spanning tree of G/e_m. Also, e_m is internally active in every spanning tree T of G, since $\mathrm{cut}(T, e_m) = \{e_m\}$, so $t_{0,j}(G) = 0$ for each j. Clearly, for $1 \leq k \leq m - 1$ the edge e_k is internally (externally) active in G with respect to T if and only if it is internally (externally) active in G/e_m with respect to T/e_m. Hence $t_{i,j}(G) = t_{i-1,j}(G/e_m)$ for $i \geq 1$. Applying the inductive hypothesis, we obtain

$$\begin{aligned} T(G; x, y) &= \sum t_{i-1,j}(G/e_m) x^i y^j \\ &= x \sum t_{i-1,j}(G/e_m) x^{i-1} y^j \\ &= x T(G/e_m; x, y) = T(G; x, y). \end{aligned}$$

(ii) Suppose e_m is a loop. Then e_m is in no spanning tree of G, and a subgraph T of G is a spanning tree of G if and only if it is a spanning tree of $G \backslash e_m$. Also e_m is externally active with respect to every spanning tree T of G since $\mathrm{cyc}(T, e_m) = \{e_m\}$. For $1 \leq k \leq m-1$ the edge e_k is internally (externally) active in G with respect to T if and only if it is internally (externally) active in

$G\backslash e_m$ with respect to the same spanning tree T. Hence $t_{i,j}(G) = t_{i,j-1}(G\backslash e_m)$, so

$$\begin{aligned}\sum_{i,j} t_{i,j}(G)x^i y^j &= y\sum_{i,j} t_{i,j-1}(G\backslash e_m)x^i y^{j-1} \\ &= yT(G\backslash e_m; x, y) = T(G; x, y).\end{aligned}$$

(iii) Suppose e_m is ordinary.

A subset T is a spanning tree of $G\backslash e_m$ if and only if it is a spanning tree of G not containing e_m. If T is a spanning tree of $G\backslash e_m$ with internal activity i and external activity j then it has the same activities as a spanning tree of G, since every other edge precedes e_m and $\mathrm{cyc}(T, e_m)$ contains an edge other than e_m.

Similarly, T is a spanning tree of G/e_m if and only $T \cup \{e_m\}$ is a spanning tree of G (no cycles in $T \cup \{e_m\}$ can be created by e_m that would not already be in T in the contraction G/e_m). If T is a spanning tree of G/e_m with internal activity i and external activity j then it has the same activities as a spanning tree of G, since every other edge precedes e_m and $\mathrm{cut}(T, e_m)$ contains an edge other than e_m since e_m is not a bridge.

It follows that $t_{i,j}(G) = t_{i,j}(G/e_m) + t_{i,j}(G\backslash e_m)$ when e_m is ordinary, and this makes the induction step go through for ordinary edges too. □

A more constructive proof that $t_{i,j}(G)$ is equal to the number of spanning trees of G of internal activity i and external activity j was given by Crapo in 1969. See for example [4, ch. 13], and also [5, X.5].

The definition of internal and external activity extends in the obvious way from spanning trees of connected graphs to maximal spanning forests of graphs more generally.

Corollary 3.15. *Let G be a graph with Tutte polynomial $T(G; x, y) = \sum t_{i,j}(G)x^i y^j$. Then $t_{i,j}(G)$ is equal to the number of maximal spanning forests of G of internal activity i and external activity j.*

Proposition 3.16. *If $|E(G)| > 0$ then $t_{0,0}(G) = 0$. If $|E(G)| > 1$ then $t_{1,0}(G) = t_{0,1}(G)$.*

Proof. If $E = \{e_1, \dots, e_m\}$ is non-empty with order $e_1 < \cdots < e_m$, then e_1 is active with respect to any maximal spanning forest F, internally if $e_1 \in F$, externally if $e_1 \notin F$. In particular, $t_{0,0}(G) = 0$.

Note that $t_{1,0}(K_2) = 1, t_{0,1}(K_2) = 0$. Asssume $m \geq 2$. If G has a least two blocks containing at least one edge then we can choose an order on E such that e_1 and e_2 belong to different blocks of G. Then e_1 and e_2 are both active with respect to every maximal spanning forest, and so $t_{1,0}(G) = 0 = t_{0,1}(G)$ in this case.

Suppose then that G is 2-connected. (If there are isolated vertices we can ignore them as the Tutte polynomial is unaffected by their presence or absence.) Let T be a spanning tree of internal activity 1 and external activity 0.

The edge e_1 is active with respect to every spanning tree, and so $e_1 \in T$. This implies $e_2 \notin T$, for otherwise e_2 would also be internally active for T ($\mathrm{cut}(T, e_2)$ cannot contain e_1, which belongs to T). So $e_1 \in \mathrm{cyc}(T, e_2)$, otherwise e_2 would be externally active.

The subgraph $T' = T - \{e_1\} \cup \{e_2\}$ is also a spanning tree of G, and has internal activity 0 and external activity 1 (the edge e_1).

Reversing the argument shows that the map $T \mapsto T'$ is a bijection between trees contributing to $t_{1,0}(G)$ and trees contributing to $t_{0,1}(G)$: if T' is a spanning tree contributing to $t_{0,1}(G)$ then $e_1 \notin T'$ but $e_2 \in T$, and interchanging e_1 and e_2 yields a spanning tree T contributing to $t_{1,0}(G)$. □

It is an easy exercise to prove Proposition 3.16 beginning with the fact that $t_{1,0}(G) = 0$ if G is not 2-connected and then inductively by deletion/contraction of an ordinary edge. However, the proof given gives more insight into why the identity holds.

The identities of Proposition 3.16 are the first of a series of identities proved by Brylawski [8]. If $|E(G)| > k$ then

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{i,j}(G) = 0.$$

Thus if $|E(G)| > 2$ then $t_{2,0}(G) - t_{1,1}(G) + t_{0,2}(G) = t_{1,0}(G)$.

The fact that $T(G; x, y)$ has degree $r(G)$ as a polynomial in x and degree $n(G)$ as a polynomial in y is immediate from the fact that $t_{i,j}(G)$ is the number of maximal spanning forests of internal activity i and external activity j. Choose the edge order $e_1 < e_2 < \cdots < e_m$ so that $e_1, \ldots, e_{r(G)}$ are the edges of a maximal spanning forest: all are internally active, and no edges are externally active when G has no loops. Or, when choosing the edge order so that $e_1, \ldots, e_{n(G)}$ are the edges in the complement of a maximal spanning forest of G, the latter having internal activity 0 provided there are no bridges, and external activity $n(G)$.

Proposition 3.17. *Let $G = (V, E)$ be a 2-connected loopless graph with Tutte polynomial $T(G; x, y) = \sum t_{i,j}(G)x^i y^j$. Then $t_{i,0}(G) > 0$ for $1 \le i \le |V| - 1$ and $t_{0,j}(G) > 0$ for $1 \le j \le |E| - |V| + 1$.*

Proof. See [5, ch. X.5]. □

3.4 Planar graphs.

Let $G = (V, E, F)$ be a connected plane graph, with set of faces F, and let $G^* = (V^*, E^*, F^*)$ be its geometric dual. To construct G^*, put a vertex in the interior of each face of G, and connect two such vertices of G^* by edges that correspond to common boundary edges between the corresponding faces of G. If there are several common boundary edges the result is a multiple edge of G^*.

We identify V^* with F, E^* with E, and F^* with V. Note that $G^{**} \cong G$.

For a spanning tree T of G, let T^ϵ denote its set of externally active edges and T^ι its set of internally active edges.

Proposition 3.18. *There is a bijection $T \mapsto T^*$ between spanning trees of G and spanning trees of G^* which switches internal and external activities. Specifically, $T^* = E \setminus T$, and $t_{i,j}(G^*) = t_{j,i}(G)$.*

Proof. The set of edges T^* in the dual G^* corresponding to the set of edges $E \setminus T$ in G together connect all the faces of G, since T has no cycles. (A cycle of edges would be required to separate one set of faces from another, their edges forming a simple closed curve partitioning the plane into inside and outside. If there are no cycles the plane remains in one piece.) Also, T^* does not contain a cycle, for otherwise it would separate some vertices in G inside the cycle from vertices outside, and this is impossible because T is spanning and its edges are disjoint from T^*.[1]

[1]Note that Euler's formula $|V| - |E| + |F| = 2$ follows from $|V(T)| = |V|, |V(T^*)| = |F|$, $|E(T)| + |E(T^*)| = |E|$ and $|E(T)| = |V(T)| - 1 = |V| - 1, |E(T^*)| = |V(T^*)| - 1 = |F| - 1$.

This shows that T^* is a spanning tree of G^*.

Given an edge $e \in T$ we have $\mathrm{cut}(T, e) = \mathrm{cyc}(T^*, e)$. Dually, given an edge $e \in E \setminus T$ we have $\mathrm{cyc}(T, e) = \mathrm{cut}(T^*, e)$. Consequently $T^\iota = (T^*)^\epsilon$ and $T^\epsilon = (T^*)^\iota$, from which it follows that $t_{i,j}(G^*) = t_{j,i}(G)$. $\square$

Corollary 3.19. *If G is a connected planar graph with dual G^* then $T(G^*; x, y) = T(G; y, x)$.*

Note that a bridge in G is a loop in G^*, a loop in G is a bridge in G^*, and that deleting (contracting) an edge in G corresponds to contracting (deleting) an edge in G^*. In other words, $(G/e)^* \cong G^* \backslash e$ and $(G\backslash e)^* \cong G^*/e$. From these properties, that $T(G^*; x, y) = T(G; y, x)$ also follows from the deletion-contraction recurrence for the Tutte polynomial.

More generally, a subgraph of G on edges $A \subseteq E$ has no cycles (i.e., is a forest) if and only if the subgraph in the dual G^* on edges $E \setminus A$ is connected. If there is a cycle in A then its edges form the boundary of a simple closed curve in the plane, inside which lies at least one vertex of G^* (corresponding to a face enclosed by the cycle) and outside of which lies another vertex of G^*. Likewise, the edges of A form a connected subgraph of G if and only if the edges of $E \setminus A$ form a forest of G^*: any cycle in G^* has to cross an edge of a connected subgraph A.

The rank and nullity functions of a planar graph and its dual are related by

$$r_{G^*}(A) = n_G(E) - n_G(E \setminus A) = |A| - r_G(E) + r_G(E \setminus A),$$

and

$$n_{G^*}(A) = r_G(E) - r_G(E \setminus A) = |A| - n_G(E) + n_G(E \setminus A).$$

Note then that $r_{G*}(E) - r_{G^*}(A) = |E \setminus A| - r_G(E \setminus A) = n_G(E \setminus A)$.[2] Thus

$$T(G; x, y) = \sum_{E \setminus A \subseteq E} (x-1)^{n_{G^*}(E \setminus A)} (y-1)^{r_{G^*}(E) - r_{G^*}(E \setminus A)} = T(G^*; y, x).$$

3.5 Spanning tree partition of subgraphs.

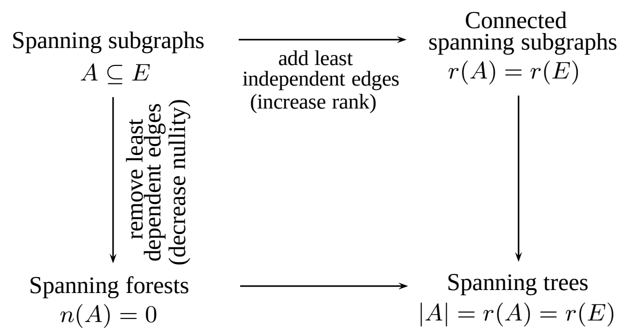
The remarks in this section rely on many facts given without proof (for which see e.g. [4, ch. 13]).

Let $G = (V, E)$ be a connected graph with a given order on its edges. For each spanning tree T of G, we have a set of externally active edges, T^ϵ, and a set of internally active edges, T^ι. The Boolean lattice of subgraphs $2^E = \{A : A \subseteq E\}$ is partitioned into Boolean intervals $[T \setminus T^\iota, T \cup T^\epsilon] = \{A : T \setminus T^\iota \subseteq A \subseteq T \cup T^\epsilon\}$ indexed by spanning trees. Given $A \subseteq E$, we have $n(A) = 0$ (i.e., $r(A) = |A|$) if and only if (V, A) is a forest, and $r(A) = r(E)$ if and only if (V, A) is a connected spanning subgraph. An edge e is *independent* of A if $r(A \cup e) = r(A) + 1$, otherwise e is dependent, and $n(A \cup e) = n(A) + 1$. Use the order on E to successively add to A the least edges $e_1, e_2, \ldots, e_{r(E)-r(A)}$ that are independent of A. This creates a connected spanning subgraph $A \cup \{e_1, \ldots, e_{r(E)-r(A)}\}$ containing A.

[2]In the terminology of the next section, an edge $e \in E \setminus A$ is independent of A in G if and only if it is a dependent edge of $E \setminus A$ in G^*. (And the dual statement holds: an edge $e \in A$ is a dependent edge of G if and only if it is an independent edge of $E \setminus A$.) The maximum number k of edges $e_1, \ldots, e_k$ such that e_i is independent of $A \cup \{e_1, \ldots, e_{i-1}\}$ for each $1 \leq i \leq k$ is equal to $r_G(E) - r_G(A)$, which is therefore equal to the maximum number k of edges $e_1, \ldots, e_k$ so that e_i is dependent on $E \setminus (A \cup \{e_1, \ldots, e_i\})$ for each $1 \leq i \leq k$, i.e., $n_{G^*}(A)$.

Similarly, given $A \subseteq E$, by removing edges dependent on A we decrease its nullity, and if $e_1, \ldots, e_{n(A)}$ are chosen to be the least such dependent edges then we obtain a unique subgraph $A \setminus \{e_1, \ldots, e_{n(A)}\}$ of nullity zero, i.e., a spanning forest of G.

If we first add least independent edges to A to make a connected spanning subgraph, and then remove least dependent edges of A we obtain a spanning tree T of G. Likewise, if we first remove the least dependent edges to make a spanning forest and then add the least independent edges we obtain (the same) spanning tree T.



This procedure locates which interval $[T \setminus T^\iota, T \cup T^\epsilon]$ the subset A belongs to. Call A an *internal subgraph* if we only need add independent edges to A in order to place it in its interval $[T \setminus T^\iota, T \cup T^\epsilon]$. In particular, (V, A) is a spanning forest and contains no externally active edges of T, i.e, $A \in [T \setminus T^\iota, T]$. (Note that A is internal in this sense if and only if it contains no broken cycle: the least edge in a cycle contributes to the external activity of the tree T containing A.)

Similarly, call A an *external subgraph* if we need only remove dependent edges from A in order to place it in $[T, T \cup T^\epsilon]$. Then (V, A) is a connected spanning subgraph containing no internally active edges of T. (If A is external, then $E \setminus A$ contains no "broken cuts".)

From the expansion $T(G; x, y) = \sum_{i,j} t_{i,j}(G) x^i y^j$ we see that $T(G; 2, 0)$ is the number of internal subgraphs (this also follows from Whitney's Broken Cycle Theorem) and $T(G; 0, 2)$ is the number of external subgraphs. Moreover, $T(G; 1, 0)$ counts the number of internal trees, and $T(G; 0, 1)$ the number of external trees.

	General	Connected	External
General	$T(G; 2, 2) = 2^{\lvert E \rvert}$	$T(G; 1, 2)$	$T(G; 0, 2)$
Forest	$T(G; 2, 1)$	$T(G; 1, 1)$	$T(G; 0, 1)$
Internal	$T(G; 2, 0)$	$T(G; 1, 0)$	$T(G; 0, 0) = 0$

(We have already seen that $T(G; 2, 0)$ counts acyclic orientations, and for a connected graph $T(G; 1, 0)$ counts acyclic orientations with unique prescribed source. See e.g. [3, Fig. 20] for an interpretation of $T(G; x, y)$ for other values of $x, y \in \{0, 1, 2\}$ in terms of orientations of G. In fact, Las Vergnas [32] gives an interpretation for $2^{i+j} t_{i,j}(G)$ in terms of orientations of G and an order on E, quoted as Theorem 25 in [12].)

Given the spanning tree partition $2^E = \bigcup_T [T \setminus T^\iota, T \cup T^\epsilon]$ of all subgraphs of

G, the subgraph expansion of the Tutte polynomial may be rewritten as follows:

$$\begin{aligned}
T(G;x,y) &= \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{n(A)}\\
&= \sum_{T}\sum_{A\in[T\setminus T^{\iota},T\cup T^{\epsilon}]}(x-1)^{|A\cap T^{\iota}|}(y-1)^{|A\cap T^{\epsilon}|}\\
&= \sum_{T}\sum_{k,\ell}\binom{|T^{\iota}|}{k}(x-1)^{k}\binom{|T^{\epsilon}|}{\ell}(y-1)^{\ell}\\
&= \sum_{T}x^{|T^{\iota}|}y^{|T^{\epsilon}|},
\end{aligned}$$

which gives Tutte's spanning tree expansion by internal and external activities.

3.6 The beta invariant.

The coefficient $t_{1,0}(G)$ is known as Crapo's *beta invariant*, or also the *chromatic invariant*, with $t_{1,0}(G) = (-1)^{|V(G)|}P'(G;1)$.

We know from the corresponding property of the chromatic polynomial that the beta invariant is unaffected by the addition or removal of parallel edges. A direct proof can be given by a deletion/contraction of a parallel edge, noting $t_{1,0}(G) = 0$ if G has a loop.

By Propositions 3.18 and 3.16, $t_{1,0}(G) = t_{1,0}(G^*)$ when G is a connected planar graph.

Two graphs are *homeomorphic* if they can both be obtained from the same graph by subdividing its edges (inserting vertices of degree 2).

Proposition 3.20. *If G and G' are homeomorphic connected graphs with at least two edges then $t_{1,0}(G) = t_{1,0}(G')$.*

NOTE. The condition on the number of edges is necessary: $t_{1,0}(K_2) = 1$ but for any path P_n on $n \geq 3$ vertices, which is homeomorphic to K_2, we have $t_{1,0}(P_n) = 0$.

Proof. Homeomorphic graphs have each some subdivision that makes them isomorphic. Hence it suffices to prove that if G' is obtained from G by subdividing an edge e into two edges e_1 and e_2 then $t_{1,0}(G) = t_{1,0}(G')$.

If e is a bridge of G then since G has another edge it is not 2-connected, so $t_{1,0}(G) = t_{1,0}(G') = 0$.

If e is not a bridge of G then e_1 is neither a bridge nor a loop of G', so $t_{1,0}(G') = t_{1,0}(G'/e_1) + t_{1,0}(G'\setminus e_1)$. As e_2 is a block of $G'\setminus e_1$ and there is another edge of G' we have $t_{1,0}(G'\setminus e_1) = 0$. Since $G'/e_1 \cong G$ this yields the desired result that $t_{1,0}(G') = t_{1,0}(G)$. □

Definition 3.21. *A* series-parallel *graph is a graph constructed from C_2 (two vertices joined by two parallel edges) by a sequence of the following two operations:*

(i) subdividing an edge (introducing a vertex of degree 2),

(ii) placing an edge parallel to an existing edge.

Series-parallel graphs are 2-connected, have no loops, and are planar.

Theorem 3.22. *Let G be a 2-connected graph with at least one edge. Then $t_{1,0}(G) \geq 1$ with equality if and only if G is series-parallel.*

Proof. If G is not 2-connected then $t_{1,0}(G) = 0$.

We prove the statement by induction on the number of edges. The base case C_2 has $T(C_2; x, y) = x + y$.

Suppose G is 2-connected with $m \geq 3$ edges and assume the truth of the assertion for 2-connected graphs with less than m edges. If G has an edge e that has been introduced in series (one of its endpoints has degree 2), then G/e is 2-connected while $G\backslash e$ is not. Hence $t_{1,0}(G\backslash e) = 0$ while by inductive hypothesis $t_{1,0}(G/e) = 1$

On the other hand, if e is parallel to another edge of G then G/e has a loop and at least one other edge and hence is not 2-connected, while $G\backslash e$ is 2-connected. By inductive hypothesis we have $t_{1,0}(G\backslash e) = 1$, so that $t_{1,0}(G) = 0 + t_{1,0}(G) = 1$.

For the converse we use the fact that a 2-connected graph G is series-parallel if and only if it contains no K_4 minor (Dirac, 1952), and that $t_{i,j}(H) \leq t_{i,j}(G)$ whenever H is a minor of a 2-connected graph G (Brylawski, [8, Corollary 6.9]). It follows in particular that $t_{1,0}(K_4) = 2 \leq t_{1,0}(G)$ whenever a 2-connected graph G is not series-parallel. □

Exercise 3.23. *Let W_n be the wheel on $n+1$ vertices (an n-cycle all of whose vertices are joined to a new central vertex). By first calculating the chromatic polynomial of W_n, find $t_{1,0}(W_n)$.*

By using $P(K_n; z) = z^{\underline{n}}$, show that $t_{1,0}(K_n) = (n-2)!$.

Proposition 3.24. *If $G = G_1 \cup G_2$ where $|V(G_1) \cap V(G_2)| = s \geq 2$ and the induced subgraph on $V(G_1) \cap V(G_2)$ is a clique K_s, then*

$$t_{1,0}(G) = t_{1,0}(G_1)t_{1,0}(G_2)/(s-2)!.$$

Note that if G has a 1-separation then it is not 2-connected and $t_{1,0}(G) = 0$.

Proof. This follows from the expression for the chromatic polynomial of a quasi-separation given in Proposition 2.22, written as

$$P(G; 1-z)P(K_s; 1-z) = P(G_1; 1-z)P(G_2; 1-z),$$

where, for connected G,

$$P(G; 1-z) = (-1)^{|V|-1}(1-z) \sum_{1 \leq i \leq |V|-1} t_{i,0}(G)z^i,$$

and the fact that $t_{1,0}(K_s) = (s-2)!$. Comparing coefficients of z^2 gives the result. □

In particular, edge-gluing a series-parallel graph to G does not change its beta invariant.

The only 3-connected graph G with beta invariant $t_{1,0}(G) = 2$ is K_4, and a similar classification of 3-connected graphs with beta invariant up to 9 has been made (see references given in [12, §7.1]). An *outerplanar* graph is a planar graph with an embedding in the plane with the property that all vertices of G lie on the outer face. A graph is outerplanar if and only if it has no K_4 minor (so it is series-parallel) or $K_{2,3}$ minor.

Theorem 3.25. [19] *If G is a simple 2-connected series-parallel graph then $t_{2,0}(G) \geq t_{0,2}(G) + 1$ with equality if and only if G is outerplanar.*

It turns out that the beta invariant $t_{1,0}(G)$ counts a certain subset of those acyclic orientations counted by $T(G; 1, 0)$ (Theorem 3.9 above).

Theorem 3.26. [Greene and Zaslavsky, 1983; Las Vergnas, 1984][3] *Let G be a connected graph and $uv \in E(G)$. The number of acyclic orientations of G with u as unique source and v as unique sink is equal to $t_{1,0}(G)$.*

Proof. Let $Q_{uv}(G)$ denote the number of acyclic orientations of G with u as unique source and v as unique sink.

Recall that $t_{1,0}(G) = 0$ if G is not 2-connected. We know that $t_{1,0}(G) = t_{1,0}(G/e) + t_{1,0}(G\backslash e)$ for an ordinary edge e, and if G has more than one edge and e is a bridge of G then $t_{1,0}(G) = 0$ (since G is not 2-connected). Also $t_{1,0}(K_2) = 1$. Finally, $t_{1,0}(G) = 0$ if G has a loop e.

When G is not 2-connected it is impossible to have an acyclic orientation of G with unique source u and unique sink v. First, if G is not connected then there are not even any acyclic orientations with unique source u, since each component has a source. Second, if G is connected with 1-separation $G_1 \cup G_2$ having $|V(G_1) \cap V(G_2)| = 1$, then an acyclic orientation restricted to G_1 has at least one source and sink, at least one of which survives as a source or sink in G. Similarly for G_2. But then there is either a source or sink in G_1 and in G_2, and these are not connected by an edge. Hence u and v are not unique as source and sink.

Clearly $Q_{uv}(K_2) = 1$ and $Q_{uv}(G) = 0$ if G has a loop.

If G has at least two edges, is 2-connected and has no loops, then G has no bridges. It remains to prove that in this case $Q_{uv}(G) = Q_{uv}(G/e) + Q_{uv}(G\backslash e)$, where e is an ordinary edge. We can choose $e = wv$ with $w \neq u, v$. In an acyclic orientation of G with unique sink v the edge wv is directed from w to v. Since u is the unique source there is at least one edge directed into w. If there is also at least one other edge directed out of w, then deleting e gives an acyclic orientation of $G\backslash e$ with unique source u and unique sink v. On the other hand, if e is the only edge directed out of w then contracting the edge e gives an acyclic orientation of G/e with unique source u and unique sink v (which is identified with w in the graph G/e). Thus partitioning acyclic orientations of G with unique source u and unique sink v according to whether or not $G\backslash wv$ is also an acyclic orientation with this property, we find that $Q_{uv}(G) = Q_{uv}(G/wv) + Q_{uv}(G\backslash wv)$. □

4 The cycle and cut space of a graph.

4.1 The incidence matrix of an oriented graph.

Let $\mathbb{F}$ be a field and $G = (V, E)$ a graph.

The *vertex space* of G over $\mathbb{F}$ is $\mathbb{F}^V = \{\mathbf{x} : V \to \mathbb{F}\}$ and the *edge space* of G is $\mathbb{F}^E = \{\mathbf{y} : E \to \mathbb{F}\}$. Our choice of notation indicates we shall think of elements of the vertex and edge spaces as column vectors $\mathbf{x} = (x_v)$ and $\mathbf{y} = (y_e)$ indexed by V and E, respectively.

If $\mathbb{F} = \mathbb{F}_q$ then we can think of an element of the vertex space as a q-colouring of G (with no restriction on it being proper). Indeed, it is possible to define vertex and edge modules over a ring $\mathbb{F}$ rather than just over a field, but this complicates the statement and proof of results that rely on cleaner properties of vector spaces over those of modules more generally. However, later when we consider k-flows of G we shall indeed take $\mathbb{F}$ to be a finite additive Abelian group

[3] The original proofs of Greene and Zaslavsky of this result and Theorem 3.9 use hyperplane arrangements. A contraction–deletion proof was given by Gebhard and Sagan [16]. Las Vergnas proved a stronger theorem in [32], giving an orientation expansion for the Tutte polynomial.

of order k (which can be made into a ring in a canonical way), and just bear in mind that the requisite results stated here for vector spaces carry *mutatis mutandis* over to modules.

The inner product of two vectors $\mathbf{y}, \mathbf{z} \in \mathbb{F}^E$ is defined by $\mathbf{y}^\top \mathbf{z} = \sum_{e \in E} y_e z_e$. (If in a field such as $\mathbb{C}$ or $\mathbb{F}_4$ we can take the Hermitian inner product, conjugating one of $\mathbf{y}$ or $\mathbf{z}$. The results that follow only use vectors such as signed indicator vectors whose entries are real/ self-conjugate.)

Let σ be a fixed orientation of G, which for each edge $e = uv$ either directs e from u to v ($u \longrightarrow v$) or from v to u ($u \longleftarrow v$). The orientation σ is required to make the definitions of the objects we shall work with, but it turns out that the results we prove are independent of the choice of σ.

For a loop e at a vertex v an orientation of e simultaneously directs v out from itself and into itself.

Definition 4.1. *The* incidence matrix *of a graph G with a given orientation of its edges is the matrix $D \in \mathbb{F}^{V \times E}$ whose (v, e)-entry is defined by*

$$D_{v,e} = \begin{cases} +1 & e \text{ is directed into } v, \\ -1 & e \text{ is directed out of } v, \\ 0 & e \text{ is not incident with } v, \text{ or } e \text{ is a loop on } v. \end{cases}$$

A loop e corresponds to a zero column of D indexed by e; each column of D indexed by an ordinary edge or bridge contains one entry $+1$, one entry -1, and remaining entries all 0.

The incidence matrix defines a linear transformation $D : \mathbb{F}^E \to \mathbb{F}^V$. For each $\mathbf{y} \in \mathbb{F}^E$,

$$(D\mathbf{y})_v = \sum_{e=uv:u \longrightarrow v} y_e - \sum_{e=uv:u \longleftarrow v} y_e.$$

This map is called the *boundary* and can be thought of as assigning the net flow of $\mathbf{y}$ to each vertex.

The transpose matrix $D^\top$ defines a linear transformation from the vertex space to the edge space: for each $x \in \mathbb{F}^V$ and $e = uv \in E$

$$(D^\top \mathbf{x})_e = \begin{cases} x_v - x_u & u \longrightarrow v \\ x_u - x_v & u \longleftarrow v. \end{cases}$$

The map $D^\top : \mathbb{F}^V \to \mathbb{F}^E$ is the *coboundary*, and assigns to each edge the difference (taken according to the edge orientation) of its endpoints in an $\mathbb{F}$-colouring of the vertices of G.

Proposition 4.2. *Let G be a graph with an orientation of its edges. Then D has rank $r(G)$ and nullity $n(G)$.*

Proof. We shall show the equivalent statement that $D^\top$ has rank $r(G)$. If $(D^\top \mathbf{x})_e = 0$ for edge $e = uv$ then $x_v - x_u = 0$, which implies $\mathbf{x}$ is constant on any connected component of G. The space of such vectors has dimension $c(G)$, whence $D^\top$ has rank $\dim(\mathbb{F}^V) - c(G) = r(G)$. It follows that D has rank $r(G)$ and nullity $\dim(\mathbb{F}^E) - r(G) = n(G)$. $\square$

The proof of Proposition 4.2 uses the fact that $\ker(D^\top)$ is the set of $\mathbb{F}$-colourings of vertices of G constant on each connected component of G, which forms a subspace of dimension $c(G)$. The orthogonal complement of the subspace $\ker(D^\top)$ (with inner product the usual dot product or Hermitian inner product) is the $r(G)$-dimensional subspace $\mathrm{im}(D)$, which consists of vectors in $\mathbb{F}^V$ whose entries sum to zero.

Let C be a cycle of G. The two possible cyclic orderings of the edges of C induce two cyclic orientations of the edges of C. Choose one of these orientations and define $\mathbf{y}_C \in \mathbb{F}^E$ by setting $(\mathbf{y}_C)_e = +1$ if e belongs to C and its cycle-orientation coincides with its orientation in G (under the fixed orientation σ chosen at the outset), $(\mathbf{y}_C)_e = -1$ if e belongs to C and its cycle-orientation is the reverse of its orientation in G, and $(\mathbf{y}_C)_e = 0$ if e does not belong to C. The vector $\mathbf{y}_C$ is called the *signed indicator vector* of C in the oriented graph G.

Proposition 4.3. *The vector space* $\ker(D)$ *has dimension* $n(G)$*. If* C *is a cycle of* G *then its signed indicator vector belongs to* $\ker(D)$*.*

Proof. That $\ker(D)$ has dimension $n(G)$ follows from Proposition 4.2. If $\mathbf{y}_C$ is the signed indicator vector of a cycle C of G then $(D\mathbf{y}_C)_v$ is 0 if v does not lie on C, and otherwise, if v belongs to edges e and f of C then the choice of sign in the definition of y_e and y_f makes $(D\mathbf{y}_C)_v$ equal to zero (e.g., if the orientation σ directs e into v and f into v then $(D\mathbf{y}_C)_v = (y_C)_e + (y_C)_f$, and $(y_C)_e = -(y_C)_f$ either way the cycle-orientation is chosen). Hence $D\mathbf{y}_C = \mathbf{0}$ when $\mathbf{y}_C$ is the signed indicator vector of a cycle C. ☐

A partition $V = V_1 \sqcup V_2$ of V into two non-empty subsets defines a *cut*, B, comprising those edges with one endpoint in V_1 and the other in V_2. We may choose one of two possible cut-orientations of B, by specifying that either all edges are direct from V_1 to V_2, or the reverse. In a similar way to cycles, define the signed indicator vector $\mathbf{y}_B$ of a cut B by setting $(\mathbf{y}_B)_e = +1$ when e belongs to B and its cut-orientation coincides with the given orientation σ of G, $(\mathbf{y}_B)_e = -1$ when e belongs to B and its cut-orientation is opposite to its orientation in G , and $(\mathbf{y}_B)_e$ when e does not belong to B.

Proposition 4.4. *The orthogonal complement of* $\ker(D)$ *is the space* $\mathrm{im}(D^\top)$ *and has dimension* $r(G)$*. If* B *is a cut in* G *then its signed indicator vector belongs to* $\mathrm{im}(D^\top)$*.*

Proof. The orthogonal complement of $\ker(D)$ has dimension $\dim(\mathbb{F}^E) - \dim(\ker(D)) = r(G)$. Given $\mathbf{y} \in \ker(D)$ and $\mathbf{x} \in \mathbb{F}^V$, we have $(D^\top\mathbf{x})^\top\mathbf{y} = \mathbf{x}^\top D\mathbf{y} = \mathbf{0}$, so that $\mathrm{im}(D^\top) \subseteq (\ker(D))^\perp$. To show that equality holds we exhibit a basis of $r(G)$ elements for $\mathrm{im}(D^\top)$.[4]

Let $\mathbf{y}_C$ be the signed indicator vector of a cycle C and $\mathbf{y}_B$ the signed indicator vector of a cut B. Suppose B is the cut formed by the partition $V_1 \sqcup V_2$ and its cut-orientation is from V_1 to V_2. Then $\mathbf{y}_B^\top\mathbf{y}_C$ is the number of edges of C going from V_1 to V_2 in its cycle-orientation, minus the number of edges going from V_2 to V_1, and this is equal to zero. Hence all signed indicator vectors of cuts belong to $(\ker(D))^\perp$.

The set of edges incident with a vertex v forms a cut with signed indicator vector the column of $D^\top$ indexed by v. If for each component of G we delete one column of $D^\top$ indexed by one of its vertices, then the remaining $|V| - c(G) = r(G)$ columns are linearly independent, and together form a basis for $\mathrm{im}(D^\top)$. ☐

[4]This step is superfluous for vector spaces where in fact it is always the case that $\mathrm{im}(D^\top) = \ker(D)^\perp$, but when $\mathbb{F}$ is an Abelian group and $\mathbb{F}^E$ is a module over $\mathbb{F}$, it in general only holds that $\mathrm{im}(D^\top) \subseteq (\ker(D))^\perp$, where orthogonal complements of modules are defined in the analagous way to vector spaces.

Proposition 4.5. *Let G be a connected graph, D its incidence matrix (for some orientation of G), and T a spanning tree of G.*

The signed indicator vectors of the cycles $\{\mathrm{cyc}(T,e) : e \in E \setminus T\}$ form a basis for the cycle space $\ker(D)$. The signed indicator vectors of the cuts $\{\mathrm{cut}(T,e) : e \in E\}$ form a basis for the cut space $\mathrm{im}(D^\top)$.

Proof. A given edge $e \in E \setminus T$ belongs to $\mathrm{cyc}(T,e)$ but no other cycle $\mathrm{cyc}(T,f)$ for $f \neq e$. Hence the signed indicator vectors of $\{\mathrm{cyc}(T,e) : e \in E \setminus T\}$ are linearly independent, and form a basis since there are $|E \setminus T| = n(G)$ of them.

Likewise, a given edge $e \in T$ belongs to $\mathrm{cut}(T,e)$ but to no other $\mathrm{cut}(T,f)$ for $f \neq e$, so the $|T| = r(G)$ signed indicator vectors of these cuts are linearly independent. □

Let G be a connected plane graph and G^* its dual. It will help to picture the edges e^* of the dual G^* to be crossing the corresponding edge e of G at right-angles. Given an orientation σ of G define the orientation σ^* of G^* by rotating the edge e oriented by σ in G clockwise by a quarter turn to give the orientation σ^* of e^*. (The edge e^* is directed from the left-hand side of e to its right-hand side.)

Proposition 4.6. *Let D denote the incidence matrix of G and D^* the incidence matrix of G^*. Then $D^*D^\top = O$. Also, $\ker(D^*) = \mathrm{im}(D^\top)$ and $\mathrm{im}((D^*)^\top) = \ker(D)$.*

Proof. Given a vertex $v \in V$ and face F incident with v, there are exactly two edges e, f belonging to F and with v as an endpoint. Then

$$(D^*D^\top)_{F,v} = (D^*)_{F,e}(D)_{v,e} + (D^*)_{F,f}(D)_{v,f}. \tag{14}$$

Note that reversing the orientation of edge e does not change the value of $(D^*)_{F,e}(D)_{v,e}$ since both signs are flipped. Likwise for reversing the orientation of e. Taking the orientation that directs e into v and f out of v (for example), we calculate that (14) is equal to $(+1)(+1) + (+1)(-1) = 0$. Hence $D^*D^\top = O$, so that $\mathrm{im}(D^*)$ is orthogonal to $\mathrm{im}(D^\top)$. Since D has rank $r(G)$ and D^* has rank $r(G^*) = n(G)$ it follows that $\mathrm{im}((D^*)^\top) = \ker(D)$ and $\ker(D^*) = \mathrm{im}(D^\top)$. □

Proposition 4.6 is an expression of the fact that cycles of G^* are cuts of G, and cuts of G^* are cycles of G.

Since faces of G correspond to vertices of G^*, another natural basis for cycles of a connected plane graph G consists of the signed indicator vectors of all but one of the face boundaries (say all but the outer face). This corresponds to the cut basis of G^* obtained by taking the signed indicator vectors of the edges incident with a common vertex, for all but one vertex of G^*.

Call a graph G^* the *abstract dual* of a graph G if $E(G) = E(G^*)$ and the minimal cuts of G^* are precisely the edge sets of cycles (minimal dependent sets) of G. This is to say that the cut space of G^* is the cycle space of G.

Theorem 4.7. (Whitney, 1933) *A graph is planar if and only if it has an abstract dual.*

Proof. See e.g. [10, ch. 4]. □

4.2 The Laplacian and the number of spanning trees.

Proposition 4.8. *Let D be the incidence matrix (with respect to some orientation) of a graph G, and let A be the adjacency matrix of G (whose (u,v)-entry is the number of edges joining u to v). Then*

$$Q = DD^\top = \Delta - A,$$

where Δ is the diagonal matrix whose (v,v)-entry is the degree of the vertex v (a loop on v contributing 2 to its degree). Consequently, Q is independent of the orientation given to G.

The matrix Q is called the *Laplacian matrix* of G.

Let $Q[u]$ denote the matrix obtained by deleting the row and column indexed by u, and $Q[u,v]$ the matrix obtained by further deleting the row and column indexed by v.

Write $Q = Q(G)$ when D is the incidence matrix of G. Note that if e is a loop then $Q(G) = Q(G \setminus e)$, since the column of the incidence matrix D of G is indexed by e is zero and contributes nothing to $DD^\top$.

Theorem 4.9. *Let G be a connected graph with Laplacian matrix Q. If u is an arbitrary vertex of G then $\det Q[u]$ is equal to the number of spanning trees of G.*

Proof. We show that $Q(G)[u]$ satisfies the same deletion–contraction recurrence as the number of spanning trees of G, $\tau(G)$, which satisfies the deletion–contraction recurrence $\tau(G) = \tau(G\backslash e) + \tau(G/e)$ when e is not a loop, and $\tau(G) = \tau(G\backslash e)$ when e is a loop. (Note that when e is a bridge, $\tau(G) = \tau(G/e)$ because $G\backslash e$ is disconnected so that $\tau(G\backslash e) = 0$.)

When e is a loop on u, $Q(G)[u] = Q(G \setminus e)[u]$.

Choose an ordinary edge $e = uv$, and let R be the $V \times V$ diagonal matrix with $R_{v,v} = 1$, and all other entries equal to 0. Then

$$Q(G)[u] = Q(G\backslash e)[u] + R,$$

from which

$$\det Q[u] = \det Q(G\backslash e)[u] + \det Q(G\backslash e)[u,v]. \tag{15}$$

Note that $Q(G\backslash e)[u,v] = Q[u,v]$. Assume in forming G/e we contract u onto v, so that $V(G/e) = V \setminus \{u\}$. Then $Q(G/e)[v]$ has rows and columns indexed by $V \setminus \{u,v\}$ with (x,y)-entry equal to $Q_{x,y}$, and so we also have that $Q(G/e)[v] = Q[u,v]$. Thus we can rewrite (15) as

$$\det Q[u] = \det Q(G\backslash e)[u] + \det Q(G/e)[v].$$

By induction $\det Q(G\backslash e)[u] = \tau(G\backslash e)$ and $\det Q(G/e)[v] = \tau(G/e)$. By the recurrence for $\tau(G)$ the result follows.

□

Since $\tau(G) = T(G;1,1)$ when G is connected, and the Tutte polynomial of an arbitrary graph is multiplicative over its connected components, Theorem 4.9 provides a polynomial-time algorithm for computing $T(G;1,1)$.

Other points (x,y) at which we already know that $T(G;x,y)$ can be computed in polynomial time in the size of G include points on the hyperbola $\{(x,y) : (x-1)(y-1) = 1\}$, where $T(G;x,y) = (x-1)^{r(E)}y^{|E|}$, and the point $(-1,0)$ (since the number of proper 2-colourings amounts to testing for bipartiteness). We shall shortly see that $T(G;0,-1)$ is computable in polynomial time (what does it count?), and later that $T(G;a,\overline{a})$ is also polynomial-time computable when a is a second, third or fourth root of unity, and $\overline{a}$ its conjuagte.

4.3 Flows and tensions

Let $G = (V, E)$ be a graph with a fixed orientation of its edges. Let $\mathbb{F}$ be a commutative ring with unity, $D : \mathbb{F}^E \to \mathbb{F}^V$ the boundary mapping, defined by the incidence matrix D of G, and $D^\top : \mathbb{F}^V \to \mathbb{F}^E$ the coboundary mapping.

Fix a maximal spanning forest T of G (comprising spanning trees of each connected component of G). We have seen in Proposition 4.5 that the signed indicator vectors of fundamental cycles $\{\mathrm{cyc}(T, e) : e \in E \setminus T\}$ form a basis for $\ker D$, and the signed indicator vectors of fundamental cuts $\{\mathrm{cut}(T, e) : e \in T\}$ form a basis for $\mathrm{im}\, D^\top$.

Let A be the additive group of the commutative ring $\mathbb{F}$. We use the notation $\mathbb{Z}_k$ for the cyclic group of order k. By the classification theorem for finite Abelian groups, A always takes the form $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_r}$ where $2 \le k_1 \mid k_2 \mid \cdots \mid k_r$ (the notation $a \mid b$ meaning that a divides b), where k_r is the least common multiple of the orders of the elements of A.

Definition 4.10. *A vector in* $\ker D$ *is called an* A-flow *of* G *and a vector in* $\mathrm{im} D^\top$ *is called an* A-tension *of* G.

An A-flow or A-tension taking values only in $B \subseteq A$ is called a B-flow or B-tension respectively.

Note that we can define A-tensions and A-flows independently of the multiplicative structure of $\mathbb{F}$; the duality between tensions and flows, however, requires that we consider A to be the additive group of some ring $\mathbb{F}$. We shall return to this duality in Section 4.3.2.

An A-tension of G corresponds to $|A|^{c(G)}$ vertex A-colourings of G: to each $\mathbf{y} \in \mathrm{im} D^\top$ corresponds $|\mathbb{F}|^{c(G)}$ vectors $\mathbf{x} \in \mathbb{F}^V$ with $D\mathbf{x} = \mathbf{y}$.

Note that there is a bijective correspondence between A-flows of G with a given orientation σ and A-flows of G with a different orientation τ: given an A-flow $\mathbf{z}$ under orientation σ, by replacing z_e by $-z_e$ for each edge e on which σ and τ differ we obtain an A-flow of G under orientation τ. A similar observation can be made for A-tensions. If $B \subseteq A$ satisfies $B = -B$ then this implies that the number of B-flows and number of B-tensions is independent of the choice of orientation of G.

The *support* of a vector $\mathbf{y} \in \mathbb{F}^E$ is defined by $\mathrm{supp}(\mathbf{y}) = \{e \in E : y_e \neq 0\}$.

A *nowhere-zero A-flow* is an A-flow with support E (in other words, an $A \setminus \{0\}$-flow). Similarly, a *nowhere-zero A-tension* is an A-tension supported on E (an $A \setminus \{0\}$-tension).

Proposition 4.11. *A graph G has a nowhere-zero $\mathbb{Z}_2$-flow if and only if G is Eulerian, and G has a nowhere-zero $\mathbb{Z}_2$-tension if and only if G is bipartite.*

4.3.1 Nowhere-zero flows

Proposition 4.12. *Let G be a cubic (i.e., 3-regular) graph. Then*

(i) G has a nowhere-zero $\mathbb{Z}_3$-flow if and only if it is bipartite.

(ii) G has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow if and only if G is edge 3-colourable.

Proof. (i) Given a nowhere-zero $\mathbb{Z}_3$-flow of G, choose the orientation of G so that the value on each edge is $+1$. Then in this orientation every vertex is either a source or sink and this yields a proper vertex 2-colouring of G. Conversely, if G has a proper 2-colouring $\mathbf{x}$ with colours $0, 1 \in \mathbb{Z}_3$ then $\mathbf{z}$ defined by $\mathbf{z} = D^\top \mathbf{x}$ (i.e., $z_e = x_v - x_u$ when e is an edge directed from u to v) is a nowhere-zero $\mathbb{Z}_3$-flow, since G is cubic.

(ii) The only way three non-zero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ can have zero sum is if they are distinct. $\square$

An Eulerian orientation of a graph G is an orientation of G with the property that the indegree at a vertex is equal to its outdegree. Clearly G must be Eulerian, and by decomposing G into an edge-disjoint union of cycles there exist Eulerian orientations of G in this case.

Proposition 4.13. *Let G be a 4-regular graph. Then there is a one-to-one correspondence between nowhere-zero $\mathbb{Z}_3$-flows of G and Eulerian orientations of G.*

Proof. For a given nowhere-zero $\mathbb{Z}_3$-flow of G, arrange the orientation σ of G so that each flow value is equal to 1. Then the only way to obtain net flow zero at a vertex is to have two edges directed out and two edges directed in. In other words, the orientation σ is Eulerian. (Put alternatively, keep the fixed orientation σ of G and for a given nowhere-zero $\mathbb{Z}_3$-flow of G preserve the orientation when flow value is $+1$ and reverse the orientation when flow value is -1: the result is an Eulerian orientation, uniquely defined by the flow values and σ.) $\square$

Theorem 4.14. *A graph with a Hamiltonian cycle (a cycle traversing all vertices of G) has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.*

Proof. First note that a Hamiltonian graph has minimum degree at least 2, and if there are any vertices of degree 2 then they can be suppressed without affecting the property of being Hamiltonian or of having a nowhere-zero flow.

Hence we may assume G has minimum degree at least 3. If all vertices have degree 3, then G is a cubic Hamiltonian graph. The Hamiltonian cycle is even so its edges can be properly coloured with two colours. The remaining edges form a perfect matching, to which a third colour can be assigned, to yield a proper edge 3-colouring of G. Now use Proposition 4.12 (ii).

This case of a cubic graph forms the basis for induction on the number of vertices of G with degree > 3.

Fix a Hamiltonian cycle C of G and a vertex v of degree $d > 3$ in G. Suppose C has edges uv and vw passing through v. The vertex v of degree $d > 3$ can be replaced by a d-cycle C_d with each vertex of C_d incident with a distinct neighbour of v, and with the further property that uw is an edge of C_d. Call the resulting graph G'. The vertices of C_d all have degree 3, so G' has fewer vertices of degree > 3 than G. The given Hamiltonian cycle C of G is transformed into a Hamiltonian cycle C' of G' by traversing C_d along the edges $C_d \setminus \{uw\}$. On the other hand, a nowhere-zero flow of G' yields a nowhere-zero flow of G (just restrict the flow values to edges not in C_d): the net flow on the d-cut of G' comprising edges incident with vertices of C_d, but not themselves contained in C_d, must equal zero.

We have the implications

(i) G Hamiltonian $\Rightarrow$ G' Hamiltonian,

(ii) G' has a nowhere-zero A-flow $\Rightarrow$ G has a nowhere-zero A-flow (for any Abelian group A).

By inductive hypothesis G', which by (i) is Hamiltonian, has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, and so we get by (ii) that G also has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. This provides the inductive step.[5] $\square$

[5] We do not need the reverse implications of (i) and (ii): do they necessarily hold?

In the early years of trying to prove the Four Colour Conjecture, Tait conjectured in 1884 that every 3-connected planar graph was Hamiltonian (an example of 2-connected planar non-Hamiltonian was known, consisting of 20 vertices and 12 pentagonal faces). Tutte in 1956 gave a counterexample with 46 vertices. (See e.g. [43] for diagrams and a succinct historical account of variations on the Four Colour Conjecture.)

The complete graph K_2 is a bridge and therefore does not have a nowhere-zero flow. K_3 is Eulerian and so has a nowhere-zero $\mathbb{Z}_2$-flow. K_4 has a proper edge 3-colouring and hence has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. On the other hand, K_4 does not have a nowhere-zero $\mathbb{Z}_3$-flow since it is a non-bipartite cubic graph and does not have a nowhere-zero $\mathbb{Z}_2$-flow since it is not Eulerian.

Proposition 4.15. *K_n has a nowhere-zero $\mathbb{Z}_2$-flow when $n \geq 3$ is odd. K_n has a nowhere-zero $\mathbb{Z}_3$-flow when $n \geq 6$ is even.*

Proof. The case of odd n follows since K_n is Eulerian. For $n = 6$ we have K_6 is the edge-disjoint union of two copies of K_3 and one copy of $K_{3,3}$. Each of these graphs has a nowhere-zero $\mathbb{Z}_3$-flow($K_{3,3}$ since it is a cubic biparitite graph). The union of these flows makes a nowhere-zero $\mathbb{Z}_3$-flow of K_6.

Consider now even $n > 6$ and assume the assertion of the theorem holds for $n-2$. The graph K_n is the edge-disjoint union of K_{n-2} and $K_{2,n}^+$, where the latter is $K_{2,n}$ with an edge e added between the vertices of degree n. By hypothesis K_{n-2} has a nowhere-zero $\mathbb{Z}_3$-flow. To make a nowhere-zero $\mathbb{Z}_3$-flow of $K_{2,n}^+$ take the sum of nowhere-zero $\mathbb{Z}_3$ flows on each of the n triangles: this is non-zero on all but possibly the edge e. If necessary, make the value on e non-zero by adding in the flow again from a single (aribitrary) triangle of edges e, e_1, e_2: this makes the value on e non-zero, and reverses the sign of the flow on e_1 and e_2. We have thus constructed a nowhere-zero $\mathbb{Z}_3$-flow of K_n. $\square$

Let $G = (V, E)$ be a graph and T a maximal spanning forest of G. Recall that for a cycle C we denote the signed indicator vector of C by $\mathbf{y}_C$.

Lemma 4.16. *Let $G = (V, E)$ be a graph and T a maximal spanning forest of G. Let A be an Abelian group and $\mathbf{b} \in A^{E \setminus T}$. Then there is a unique A-flow $\mathbf{z}$ of G such that $z_e = b_e$.*

Proof. The vector

$$\mathbf{z} = \sum_{e \in E \setminus T} b_e \mathbf{y}_{\mathrm{cyc}(T,e)}$$

as a linear combination of basis vectors for $\ker D$ is an A-flow and since $e \notin \mathrm{cyc}(T, f)$ when $f \neq e$ the value of $\mathbf{z}$ at e is given by $z_e = b_e$. Conversely, if an A-flow takes value b_e at each $e \in E \setminus T$ then it is equal to $\mathbf{z}$ as defined above, since any vector has a unique expression as a linear combination of basis vectors. $\square$

Theorem 4.17. (Tutte, [46].) *Let A be a finite Abelian group of order k and G a graph with an orientation of its edges. Then the number of nowhere-zero A-flows of G is*

$$F(G; k) = \sum_{F \subseteq E} (-1)^{|E|-|F|} k^{n(F)}.$$

Proof. By Lemma 4.16 the number of A-flows of any subgraph (V, F) of $G = (V, E)$ is equal to $k^{|F|-r(F)}$, since a maximal spanning forest of (V, F) has $r(F)$ edges. Equivalently, $k^{n(F)}$ is the number of A-flows of G whose support is contained in F. The result follows by the inclusion-exclusion principle. $\square$

The polynomial $F(G;k)$ is called the *flow polynomial* of G.

Proposition 4.18. *The flow polynomial satisfies*

$$F(G;k) = \begin{cases} F(G/e;k) - F(G\backslash e;k) & e \textit{ ordinary,} \\ 0 & e \textit{ a bridge,} \\ (k-1)F(G\backslash e) & e \textit{ a loop,} \\ 1 & E = \emptyset. \end{cases}$$

Proof. When $E = \emptyset$ the subgraph expansion for $F(G;k)$ gives $F(G;k) = 1$. When G has a bridge e it does not have a nowhere-zero flow, for $\{e\}$ is a cut of G. If e is a loop, on the other hand, then we can freely assign any non-zero value to it and still have a nowhere-zero flow. When e is ordinary, we have a bijection between nowhere-zero flows of $G\backslash e$ and flows of G that are zero only at e, and between nowhere-zero flows of G/e and flows of G that are nowhere-zero except possibly at e. (This argument also works when e is a bridge, but it needs to be shown that in this case $F(G\backslash e;k) = F(G/e;k)$, which amounts to showing that $F(G;k) = 0$.) ◻

We know that nowhere-zero A-tensions are counted by $|A|^{-c(G)}P(G;|A|) = (-1)^{r(G)}T(G;1-|A|,0)$, and by the duality between tensions and flows for a planar graph the number of nowhere-zero A-flows is counted by $(-1)^{r(G^*)}T(G^*;1-|A|,0) = (-1)^{n(G)}T(G;0,1-|A|)$.

Corollary 4.19. *The flow polynomial is given by*

$$F(G;k) = (-1)^{n(G)}T(G;0,1-k).$$

Proof. Use Proposition 4.18 and the "Recipe Theorem" (Theorem 3.6). ◻

It follows in particular that $(-1)^{n(G)}T(G;0,-1)$ is equal to 1 if G is Eulerian and equal to 0 otherwise.

Note that we have a lemma entirely analogous to Lemma 4.16 for tensions:

Lemma 4.20. *Let $G = (V,E)$ be a graph and T a maximal spanning forest of G. Let A be an Abelian group and $\mathbf{a} \in A^T$. Then there is a unique A-tension $\mathbf{y}$ of G such that $y_e = a_e$.*

Proof. Take the vector $\mathbf{y} = \sum_{e\in T} a_e \mathbf{y}_{\mathrm{cut}(T,e)}$. ◻

If A has order k, Lemma 4.20 implies the number of A-tensions of a spanning subgraph (V,F) is equal to $k^{r(F)}$, so that the number of A-tensions of $G=(V,E)$ with support contained in F is equal to $k^{r(F)}$. The inclusion-exclusion principle then yields the following subgraph expansion for the number of A-tensions of G:

$$k^{-c(G)}P(G;k) = \sum_{F\subseteq E} (-1)^{|E|-|F|}k^{r(F)}.$$

(Compare with the spanning subgraph expansion we obtained by inclusion-exclusion for $P(G;z)$ in Theorem 2.23.)

We have seen (Theorem 4.17) that the existence of an A-flow does not depend on the structure of A, only its order, i.e., if A and A' are Abelian groups with $|A| = |A'|$ then G has a nowhere-zero A flow if and only if G has a nowhere-zero A'-flow. What is not perhaps yet apparent is whether the existence of a nowhere-zero A-flow implies the existence of a nowhere-zero A'-flow when $|A'| > |A|$. (Thinking of A-flows as duals of A-tensions, it is obvious that if G

has a nowhere-zero A-tension then it has a nowhere-zero A'-tension, by using the correspondence of nowhere-zero A-tensions with proper A-colourings.)

Let $k \in \mathbb{Z}$, $k \geq 2$, and G a graph with an orientation of its edges. A *nowhere-zero k-flow* of G is a $\mathbb{Z}$-flow $\mathbf{z}$ of G such that $0 < |\mathbf{z}_e| < k$ for each edge e. It is clear that a nowhere-zero k-flow yields a nowhere-zero $\mathbb{Z}_k$-flow, by reading the integer flow values modulo k. The converse is not in general true: a nowhere-zero $\mathbb{Z}_k$-flow may not yield a nowhere-zero k-flow (some boundary values may be non-zero multiples of k). However, it turns out that if there are nowhere-zero $\mathbb{Z}_k$-flows then there is at least one nowhere-zero k-flow of G.

Theorem 4.21. (Tutte, 1950) *A graph has a nowhere-zero $\mathbb{Z}_k$-flow if and only if it has a nowhere-zero k-flow.*

Proof. See e.g. [25], [39], [10]. □

Kochol [31] shows that the number of nowhere-zero k-flows is also a polynomial in k (not the same as the flow polynomial $F(G;k)$).

To summarize, we have the following "Equivalence Theorem":

Theorem 4.22. *Let G be a graph with orientaion of its edges. For every integer $k \geq 2$ the following are equivalent:*

(i) for some Abelian group of order k, G has a nowhere-zero A-flow;

(ii) for every Abelian group of order k, G has a nowhere-zero A-flow;

(iii) G has a nowhere-zero k-flow.

For any Abelian group A there are loopless graphs G that do not have a nowhere-zero A-tension (take $|A| > \chi(G)$). The situation for nowhere-zero A-flows is quite different.

A long-standing open problem is the 5-flow conjecture:

Conjecture 4.23. (Tutte, 1954) *Every bridgeless graph has a nowhere-zero 5-flow.*

A connected graph G is t-edge-connected if deleting a subset of $< t$ edges does not disconnect G; in other words, any cut of G has at least k edges. A bridgeless graph is 2-edge-connected.

Suppose G is a minimal graph with respect to the property of having a nowhere-zero k-flow, in that deleting or contracting an edge of G yields a graph that has a nowhere-zero k-flow. Then G must be 3-edge-connected. For suppose there are two edges e_1 and e_2 forming a cut of G. By hypothesis G/e_1 has a nowhere-zero k-flow $\mathbf{z}$. But $\mathbf{z}$ can be extended to a nowhere-zero k-flow of G by setting $z_{e_2} = \pm z_{e_1}$, the sign chosen according to the orientation of e_2 relative to that of e_1 (the net flow on the cut $\{e_1, e_2\}$ must equal zero). Hence there are no 2-cuts of G.

We shall sketch an early result moving towards Conjecture 4.23, for which the following lemma is key.

Lemma 4.24. *Every 3-edge-connected graph has three spanning trees T_1, T_2, T_3 with $T_1 \cap T_2 \cap T_3 = \emptyset$.*

Proof. See [25], [39]. □

Theorem 4.25. (Jaeger, 1978; Kilpatrick, 1978) *Every bridgeless graph has a nowhere-zero 8-flow.*

Proof. We choose $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let G be a 3-edge-connected graph. Given three spanning trees T_1, T_2, T_3 of G with empty intersection, for $i = 1, 2, 3$ construct $\mathbb{Z}_2$-flows $\mathbf{z}_i$ of G whose supports contain $E \setminus T_i$ (using Lemma 4.16). Then $(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z_3})$ is a $\mathbb{Z}_2^3$-flow of G which is nowhere-zero precisely because $T_1 \cap T_2 \cap T_3 = \emptyset$. □

A little later the 8-flow theorem was improved to within one of Tutte's 5-flow conjecture:

Theorem 4.26. (Seymour, 1981) *Every bridgeless graph has a nowhere-zero* 6*-flow.*

Proof. The proof is outlined e.g. in [25], [39] and [10]. It involves the reduction of a given 3-edge-connected graph G by contraction of cycles to a graph G' that is shown to have a nowhere-zero 3-flow. This reduction is then used to lift the $\mathbb{Z}_3$-flow of G' to a $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow of G. □

The 5-flow Conjecture if true will be best possible, because there are graphs that do not have a nowhere-zero 4-flow.

Proposition 4.27. *The Petersen graph P does not have a proper edge 3-colouring.*

Proof. Picture the usual drawing of the Petersen graph as an outer 5-cycle C_0 and an inner 5-cycle C_1 formed by joining vertices cyclically two apart, and a matching joining corresponding vertices on the two cycles. Suppose for a contradiction that there is a proper edge 3-colouring of P with colours a, b, c. In a proper edge 3-colouring of a cubic graph each colour must appear exactly once at each vertex. Since C_0 has odd length, each edge colour appears at least once on it. Suppose edge uv on the outer cycle C_0 is coloured a and that ux and vy are edges of the matching joining C_0 to C_1. Then both ux and vy cannot be coloured a, and this implies there are two edges of C_1 (one with endpoint x, the other with endpoint y) that receive colour a. Since the same argument applies to each of the other colours b and c, this leads to the impossible conclusion that each colour appears twice on the inner cycle C_1. □

Propositions 4.12 and 4.27 together imply that the Petersen graph does not have a nowhere-zero 4-flow. A cubic graph that does not have a nowhere-zero 4-flow (equivalently, a proper edge 3-colouring) is a *snark*. The Four Colour Theorem is equivalent to the assertion that no snark is planar (see Theorem 4.33 below).

Tutte made a further conjecture, that the Petersen graph was the only obstacle to having a nowhere-zero 4-flow, i.e., that all snarks contain the Petersen graph a a minor:

Conjecture 4.28. (Tutte, 1966) *Every bridgeless graph with no subgraph contractible to the Petersen graph has a nowhere-zero* 4*-flow.*

In 1980 Walton and Welsh proved, using the Four Colour Theorem, that every bridgeless graph with no subgraph contractible to $K_{3,3}$ has a nowhere-zero 4-flow.

A theorem of Grötzsch states that every loopless planar graph without triangles has a proper vertex 3-colouring. By duality, this is to say that every bridgeless planar graph without 3-cuts has a nowhere-zero 3-flow. Tutte was led to add to his series of conjectures with:

Conjecture 4.29. *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

In 1976 Jaeger proved that every 4-edge-connected graph has a nowhere-zero 4-flow, using the fact that such graphs always have a pair of edge-disjoint spanning trees to construct a pair of $\mathbb{Z}_2$-flows supported on their complements, and then to piece these together to make a nowhere-zero $\mathbb{Z}_2^2$-flow.

4.3.2 Duality between tensions and flows

Considering A as the additive group of $\mathbb{F}$, we have seen in Proposition 4.4 that A-flows and A-tensions are orthogonal complements: if $\mathbf{y} \in \mathrm{im} D^\top$ and $\mathbf{z} \in \ker D$ then $\mathbf{y}^\top \mathbf{z} = 0$, where the multiplication of elements of A is performed in $\mathbb{F}$. This relation of orthogonal complementarity depends on the ring $\mathbb{F}$ satisfying a certain technical condition, namely, that it has a generating character. It is not important to know what this means. For our purposes it suffices to know that this technical condition on $\mathbb{F}$ is satisfied when we take it to be the ring $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2} \oplus \cdots \oplus \mathbb{Z}_{k_r}$ (additive group $A \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_r}$) or the finite field $\mathbb{F}_{p^r}$ (additive group $A \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$). Since A is isomorphic to $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_r}$ for some $k_1, k_2, \ldots, k_r$, we can always assume the ring $\mathbb{F}$ is chosen so that A-tensions and A-flows are orthogonal complements. (Subject to this condition, the multiplicative structure of $\mathbb{F}$ is immaterial, so for example we can take $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ with component-wise multiplication, or with multiplication making it isomorphic to a finite field.)

In particular, for planar graphs G an A-tension of G is an A-flow of G^*, and an A-flow of G is an A-tension of G (Proposition 4.6). When G is embedded in the plane, this establishes a correpondence between face A-colourings of G and A-flows of G (the latter are A-tensions of G^*).

Proposition 4.30. *Let A be an Abelian group of order k. Then a plane graph G has a proper k-colouring if and only if its dual G^* has a nowhere-zero A-flow.*

In fact, for a general graph $G = (V, E)$ 2-cell embedded in an orientable surface (meaning every face is homeomorphic to an open disk), there is a correspondence between nowhere-zero A flows of G and proper face A-colourings of G. The orientation of edges of G determines for each edge e a face on the left and a face on the right. For each edge e, let a_e be the colour on the face on the left of e, and b_e the colour on the face on the right. Then the vector $\mathbf{a} - \mathbf{b} \in A^E$ is an A-flow of G.

Proposition 4.31. *A graph G has a nowhere-zero $\mathbb{Z}_k$-flow if it has a 2-cell embedding in some orientable surface that is face k-colourable.*

Returning to the case of planar graphs, here is an illustration where the structure of A as Abelian group has an impact on the nature of nowhere-zero A-flows as a set. (We have seen that the *number* of nowhere-zero A-flows is independent of the structure of A.)[6]

Proposition 4.32. *Every planar graph has a proper 4-colouring if and only if every planar graph is the union of two if its Eulerian subgraphs.*

Proof. By Proposition 4.30 a plane graph has a proper 4-colouring if and only if its dual has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. In a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow of a graph G,

[6] This is phenomenon is reminiscent of spanning tree activities (the set of spanning trees with given internal and external activity depends on the order imposed on E, but the size of this set is independent of this order). Sometimes, though, a particular choice of structure for A can help prove a result about the number of nowhere-zero A-flows, as happens here, just as for spanning tree activities a partiicular choice for the order on E helped prove such facts as $t_{1,0}(G) = t_{0,1}(G)$.

the first component has support an Eulerian subgraph G_1 of G, and the second component also has support an Eulerian subgraph G_2 of G. If the $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow is nowhere-zero then each edge of G belongs to either G_1 or to G_2. $\square$

If we had chosen $A = \mathbb{Z}_4$ instead in the proof of Proposition 4.32 then the correspondence of nowhere-zero A-flows with pairs of Eulerian subgraphs would not have been so evident.

Theorem 4.33. (Tait, 1880) *Every planar graph has a proper face 4-colouring if and only if every planar cubic graph has a proper edge 3-colouring.*

Proof. All plane graphs are face 4-colourable if and only if all plane graphs are vertex 4-colourable. A given plane graph can be made into a triangulation (all faces triangles, including the outer face) by triangulating each face by the addition of chords. Hence every plane graph is vertex 4-colourable if and only if every plane triangulation is vertex 4-colourable. The dual of a plane triangulation is a plane cubic (i.e., 3-regular) graph. Hence, every plane triangulation is vertex 4-colourable if and only if every plane cubic graph is face 4-colourable. A plane graph is face 4-colourable if and only if it has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. A nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow of a cubic graph must assign each of the three non-zero group elements to the three edges incident with a given vertex. In other words, a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow of a cubic graph is a proper edge 3-colouring. $\square$

Theorem 4.34. (Heawood, 1890) *A plane triangulation has a proper vertex 3-colouring if and only if it is Eulerian.*

Proof. The dual G^* of a plane triangulation G is a cubic plane graph, and G^* has a proper face 3-colouring if and only if it has a nowhere-zero $\mathbb{Z}_3$-flow. A cubic graph has a nowhere-zero $\mathbb{Z}_3$-flow if and only if it is bipartite (Proposition 4.12 (i)). G^* is bipartite if and only if G is Eulerian. $\square$

4.3.3 Hamming weight enumerator for tensions and flows

Let $G = (V, E)$ be a graph, A an Abelian group of order k, and $\mathcal{C}$ the set of A-flows of G and its orthogonal complement $\mathcal{C}^\perp$ the set of A-tensions of G.

The monochrome polynomial $B(G; k, y)$ of G was defined in Proposition 2.29 in terms of vertex k-colourings, but we can write it in terms of A-tensions as follows:

$$k^{-c(G)} B(G; k, y) = \sum_{\mathbf{z} \in \mathcal{C}^\perp} y^{|E| - |\mathrm{supp}(\mathbf{z})|}. \tag{16}$$

In coding theory $|\mathrm{supp}(\mathbf{z})|$ is called the *Hamming weight* of the vector $\mathbf{z}$ and the polynomial on the right-hand side of (16) is known as the (Hamming) *weight enumerator* of the code $\mathcal{C}^\perp$.

By deletion-contraction and the Recipe Theorem we have seen that

$$B(G; k, y) = k^{c(G)} (y-1)^{r(G)} T(G; \frac{y-1+k}{y-1}, y). \tag{17}$$

A code over a field $\mathbb{F}$ is a special type of matroid, namely one that is representable over $\mathbb{F}$. The point $(\frac{y-1+k}{y-1}, y)$ lies on the hyperbola $(x-1)(y-1) = k$. Greene [20] was first to make the connection between the Tutte polynomial and linear codes over a field of k elements, proving that the Tutte polynomial of the matroid of a code specializes on the hyperbola $(x-1)(y-1) = k$ to the weight enumerator of the code (effectively, identity (17) generalized to codes/representable matroids).

The dual version of the monochrome polynomial (the weight enumerator for A-tensions (16)) is the weight enumerator for A-flows:

$$C(G;k,x) = \sum_{\mathbf{z}\in\mathcal{C}} x^{|E|-|\mathrm{supp}(\mathbf{z})|} = (x-1)^{n(G)} T(G;x,\frac{x-1+k}{x-1}). \qquad (18)$$

(This identity can be proved by an inductive deletion-contraction argument, as for the monochrome polynomial.) Thus by identities (17) and (18) we have

$$B(G;k,y) = k^{|V(G)|-|E(G)|}(y-1)^{|E(G)|} C(G;k,\frac{y-1+k}{y-1}), \qquad (19)$$

which amounts to MacWilliams identity in coding theory.

4.4 Bicycles

In this section we take $\mathbb{F} = \mathbb{F}_2$ in the incidence mapping $D : \mathbb{F}^E \to \mathbb{F}^V$ for a graph $G = (V, E)$. We let $\mathcal{C} = \ker D$ and so $\mathcal{C}^\perp = \mathrm{im}\, D^\top$. Vectors in $\mathcal{C}$ are indicator vectors of *Eulerian subgraphs* of G (sometimes just called cycles – although we shall reserve the term cycle for a connected 2-regular subgraph – or *even subgraphs* of G). Vectors in $\mathcal{C}^\top$ are indicator vectors of (edge) *cuts* of G. (A 2-tension has support a cut, equal for some $V_0 \subseteq V, V_1 = V \setminus V_0$ to the set of edges with one endpoint in V_0 and the other in V_1.)

An Eulerian subgraph meets a cut in an even number of edges (by orthogonality of flows and tensions, and by definition when considering cuts comprising edges from $\{v\}$ to $V \setminus \{v\}$, these vertex-cuts together spanning all cuts).

We identify a subset of edges of G with its indicator vector.

A vector $\mathbf{x}$ in the intersection $\mathcal{C} \cap \mathcal{C}^\perp$ is called a *bicycle* of G, and is self-orthogonal, i.e., $\mathbf{x}^\top\mathbf{x} = 0$. So a bicycle has an even number of edges.

A bicycle is an Eulerian subgraph that meets every other Eulerian subgraph in an even number of edges (as well as every cut in an even number of edges). Alternatively, a bicycle is a cut that meets every other cut in an even number of edges (as well as meeting every Eulerian subgraph in an even number of edges).

In short, a bicycle is a cutset that is also an Eulerian subgraph of G. In particular, if G is itself a bipartite Eulerian graph then E (the all-one vector) is a bicycle.

For more about bicycles see Sections 14.15-16 and 15.7 in [18] (from which the material in this section is adapted), and for the usefulness of bicycles in relation to knots see Chapter 17 of the same reference.

Theorem 4.35. *Let e be the edge of a graph G. Then precisely one of the following holds:*

(i) e belongs to a bicycle,

(ii) e belongs to a cut B such that $B \setminus \{e\}$ is Eulerian,

(iii) e belongs to an Eulerian subgraph C such that $C \setminus \{e\}$ is a cut.

Proof. Suppose $e \in E(G)$ and $\mathbf{e}$ is its indicator vector in $\mathbb{F}_2^E$. If e belongs to a bicycle with indicator vector $\mathbf{x}$ then $\mathbf{x}^\top\mathbf{e} \neq 0$ and therefore $e \notin (\mathcal{C} \cap \mathcal{C}^\perp)^\perp = \mathcal{C}^\perp + \mathcal{C}$. If e does not belong to a bicycle then $\mathbf{e}$ is orthogonal to all vectors in $\mathcal{C} \cap \mathcal{C}^\perp$ and so $\mathbf{e} \in \mathcal{C} + \mathcal{C}^\perp$. In other words, e is either contained in a bicycle or e is the symmetric difference of an Eulerian subgraph and a cutset.

In any representation of e as the symmetric difference of an Eulerian subgraph and a cut, either e will always belong to the Eulerian subgraph, or e will always belong to the cut. For suppose that $\mathbf{e} = \mathbf{z} + \mathbf{y} = \mathbf{z}' + \mathbf{y}'$ where $\mathbf{z}, \mathbf{z}' \in \mathcal{C}$ and $\mathbf{y}, \mathbf{y}' \in \mathcal{C}^{\perp}$. Then $\mathbf{z} + \mathbf{z}' \in \mathcal{C}$ and $\mathbf{y} + \mathbf{y}' \in \mathcal{C}^{\perp}$ so $\mathbf{z} + \mathbf{z}' = \mathbf{y} + \mathbf{y}'$ is a bicycle. Since e does not belong to a bicycle, it must belong to both or neither of $\mathbf{z}$ and $\mathbf{z}'$,and to neither or both of $\mathbf{y}$ and $\mathbf{y}'$, respectively (since $\mathbf{e} = \mathbf{z} + \mathbf{y}$). □

An edge e of G is of *bicycle-type*, *cut-type* or *flow-type* according as (i), (ii) or (iii) holds in the statement of Theorem 4.35, respectively. This is known as the principal tripartition of the edges of G.

A bridge is an edge of cut-type [take cut $B = \{e\}$ in (ii)] and a loop is an edge of flow-type [take Eulerian subgraph $\{e\}$ in (iii)].

If G is planar then edges of bicycle-type in G remain of bicycle-type in G^*. By flow–tension duality, edges of cut-type in G are edges of flow-type in G^*, and similarly edges of flow-type in G^* are edges of cut-type in G^*.

See [18, Theorem 14.16.2] for a simple polynomial-time algorithm, involving the Laplacian matrix $DD^{\top}$, to decide what type an edge has in the principal tripartition.

Lemma 4.36. *Let G be a graph with bicycle space of dimension d, and e an edge of G. The following table gives the dimension of the bicycle space of G/e and $G\backslash e$.*

Type of e	G/e	$G\backslash e$
Bridge or loop	d	d
Bicycle-type	$d-1$	$d-1$
Cut-type, not bridge	d	$d+1$
Flow-type, not loop	$d+1$	d

Proof. A bridge belongs to no cycle and hence to no Eulerian subgraph, and therefore to no bicycle. So any bicycle of G is a bicycle of $G\backslash e$. Conversely, a bicycle of $G\backslash e$ is also a bicycle of G. Likewise, bicycles of G/e correspond to bicycles of G.

Similarly, a loop belongs to no cut and hence to no bicycle, so bicycles of G are bicycles of $G\backslash e$, and conversely. For a loop we have $G/e \cong G\backslash e$.

For an ordinary edge e we shall find the following two observations useful:

(i) If e is not a loop and belongs an Eulerian subgraph C, then $C \setminus \{e\}$ is neither an Eulerian subgraph of G nor of $G\backslash e$. On the other hand, $C \setminus \{e\}$ is an Eulerian subgraph of G/e.

(ii) Dually, if e is not a bridge and belongs to a cut B, then $B \setminus \{e\}$ is neither a cut of G nor of G/e. On the other hand, $B \setminus \{e\}$ is a cut of $G\backslash e$.

Suppose then that e is an ordinary edge. We distinguish the three cases of the principal tripartition:

(a) e belongs to a bicycle A.

By (i) and (ii), $A \setminus \{e\}$ is not a bicycle of G, $G\backslash e$ or G/e. On the other hand, any bicycle of G which does not contain e remains a bicycle of $G\backslash e$ and G/e. Hence the bicycle spaces of $G\backslash e$ and of G/e both correspond to the subspace of bicycles of G that do not contain e, and their dimensions are therefore 1 less than the bicycle dimension of G.

(b) e belongs to a cut B, such that $B \setminus \{e\}$ is an Eulerian subgraph of G.

By (ii), the set $B \setminus \{e\}$ is a cut of $G\backslash e$, but not of G or G/e. Hence $B \setminus \{e\}$ is a bicycle of $G\backslash e$, but not of G or G/e. The effect is to increase the dimension of the bicycle space of $G\backslash e$ by 1. All bicycles of G are bicycles of $G\backslash e$ since e is of cut-type, and so bicycles of $G\backslash e$ are bicycles of G together with symmetric difference of bicycles of G with the fixed set $B \setminus \{e\}$. On the other hand, the dimension of the bicycle space of G/e coincides with that of G, all bicycles of G being bicycles of G/e, and no others.

(c) e belongs to an Eulerian subgraph C such that $C \setminus \{e\}$ is a cut.

By (i), the set $C \setminus \{e\}$ is an Eulerian subgraph of G/e, but not of G or $G\backslash e$. Hence $C \setminus \{e\}$ is a bicycle of G/e, but not of G or $G\backslash e$. Similarly to case (b), this implies the dimension of the bicycle space of G/e is 1 more than that of G, while $G\backslash e$ has the same bicycle dimension as G.

□

Lemma 4.37. *Let $G = (V, E)$ be a graph with bicycle space of dimension $b(G)$, and let e be an edge of G. Then the graph invariant*

$$f(G) = (-1)^{|E|}(-2)^{b(G)}$$

satisfies

$$f(G) = \begin{cases} (-1)f(G/e) & e \text{ a bridge,} \\ (-1)f(G\backslash e) & e \text{ a loop,} \\ f(G/e) + f(G\backslash e) & e \text{ ordinary.} \end{cases}$$

Proof. We use Lemma 4.36.

If e is a bridge or loop then the bicycle spaces of G/e, $G\backslash e$ and G are all of the same dimension, and this implies the first two cases.

Suppose e is ordinary. If e is of cut-type then

$$\begin{aligned} f(G/e) + f(G\backslash e) &= (-1)^{|E|-1}(-2)^{b(G)} + (-1)^{|E|-1}(-2)^{b(G)+1} \\ &= (-1)^{|E|}(-2)^{b(G)}. \end{aligned}$$

If e is of flow-type then

$$\begin{aligned} f(G/e) + f(G\backslash e) &= (-1)^{|E|-1}(-2)^{b(G)+1} + (-1)^{|E|-1}(-2)^{b(G)} \\ &= (-1)^{|E|}(-2)^{b(G)}. \end{aligned}$$

If e belongs to a bicycle then

$$f(G/e) + f(G\backslash e) = 2(-1)^{|E|-1}(-2)^{b(G)-1} = (-1)^{|E|}(-2)^{b(G)}.$$

□

By the Recipe Theorem (Theorem 3.6) we obtain:

Theorem 4.38 ([42])**.** *Let $G = (V, E)$ be a graph and let $b(G)$ denote the dimension of its bicycle space. Then $(-1)^{|E|}(-2)^{b(G)} = T(G; -1, -1)$.*

Corollary 4.39. *A connected graph G has no non-trivial bicycles if and only if G has an odd number of spanning trees.*

Proof. We have $T(G; -1, -1) \equiv T(G; 1, 1) \pmod 2$. □

4.5 $\mathbb{Z}_3$-tension-flows

In this section we take $\mathbb{F} = \mathbb{F}_3$, whose additive group is isomorphic to $\mathbb{Z}_3$, and consider the intersection of the space of $\mathbb{Z}_3$-flows and the space of $\mathbb{Z}_3$-tensions. If $D : \mathbb{F}_3^E \to \mathbb{F}_3^V$ is the incidence mapping, and we let $\mathcal{C} = \ker D$, so that $\mathcal{C}^\perp = \operatorname{im} D^\top$, then we shall call a vector in $\mathcal{C} \cap \mathcal{C}^\perp$ a $\mathbb{Z}_3$*-tension-flow*. In other words, a $\mathbb{Z}_3$-tension-flow is both a $\mathbb{Z}_3$-tension and a $\mathbb{Z}_3$-flow, and is self-orthogonal in $\mathbb{F}_3^E$. (In this terminology we could have called bicycles $\mathbb{Z}_2$-*tension-flows*.)

Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity. In [26] Jaeger proved by a deletion-contraction argument that $T(G; \omega, \omega^2) = \pm \omega^{|E| + \dim \mathcal{C}} (i\sqrt{3})^{\dim(\mathcal{C} \cap \mathcal{C}^\perp)}$, using the *principal quadripartition* of the edges of a graph (a generalization to flows and tensions over finite fields of characteristic $\neq 2$ of the principal tripartition). Gioan and Las Vergnas [17] provide a linear algebra proof that has the benefit of determining the sign. It is this latter proof that we shall present here.

Recall that we say vectors $\mathbf{y}$ and $\mathbf{z}$ are orthogonal if $\mathbf{y}^\top \mathbf{z} = 0$. A *self-orthogonal* vector (also called an *isotropic* vector) is a vector $\mathbf{z}$ with $\mathbf{z}^\top \mathbf{z} = 0$.

Lemma 4.40. *Let $\mathcal{C}$ be a finite-dimensional vector space over a field of characteristic not equal to 2. Then $\mathcal{C}$ has an orthogonal basis.*

Proof. Let $\{\mathbf{z}_1, \ldots, \mathbf{z}_d\}$ be a basis for $\mathcal{C}$. If there is an index $1 \leq i \leq d$ such that $\mathbf{z}_i$ is not self-orthogonal then reindex in such a way that $i = 1$ and set $\mathbf{z}_1' = \mathbf{z}_1$. Otherwise, if there is an index $2 \leq i \leq d$ such that $\mathbf{z}_1 + \mathbf{z}_i$ is not self-orthogonal then set $\mathbf{z}_1' = \mathbf{z}_1 + \mathbf{z}_i$. In both cases update $\mathbf{z}_j$ as $\mathbf{z}_j - \frac{\mathbf{z}_1'^\top \mathbf{z}_j}{\mathbf{z}_1'^\top \mathbf{z}_1'} \mathbf{z}_1'$ for $2 \leq j \leq d$. Now $\mathbf{z}_1'$ and $\mathbf{z_j}$ are orthogonal for $2 \leq j \leq d$.

Otherwise the vectors $\mathbf{z}_j$ are self-orthogonal for $1 \leq j \leq d$, and $\mathbf{z}_1 + \mathbf{z}_j$ is self-orthogonal for $2 \leq j \leq d$. The latter implies $\mathbf{z}_1^\top \mathbf{z}_1 + 2\mathbf{z}_1 \mathbf{z}_j + \mathbf{z}_j^\top \mathbf{z}_j = 2\mathbf{z}_1^\top \mathbf{z}_j = 0$. Hence $\mathbf{z}_1^\top \mathbf{z}_j = 0$ in characteristic $\neq 2$. Set $\mathbf{z}_1' = \mathbf{z}_1$.

In all three cases $\mathbf{z}_1', \mathbf{z}_2, \ldots, \mathbf{z}_d$ comprise a basis of $\mathcal{C}$ such that $\mathbf{z}_1'$ is orthogonal to the space generated by the remaining vectors $\mathbf{z}_2, \ldots, \mathbf{z}_d$.

The result now follows by induction. ☐

Lemma 4.41. *The self-orthogonal vectors of an orthogonal basis of $\mathcal{C}$ form a basis for $\mathcal{C} \cap \mathcal{C}^\perp$.*

Proof. Let $\mathbf{z}_1, \ldots, \mathbf{z}_d$ form an orthogonal basis for $\mathcal{C}$, and $\mathbf{z} = \sum_{1 \leq j \leq d} a_j \mathbf{z}_j \in \mathcal{C} \cap \mathcal{C}^\perp$. For $1 \leq i \leq d$ we have $0 = \mathbf{z}^\top \mathbf{z}_i = \sum_{1 \leq j \leq d} a_j \mathbf{z}_j^\top \mathbf{z}_i = a_i \mathbf{z}_i^\top \mathbf{z}_i$. Hence if $\mathbf{z}_i^\top \mathbf{z}_i \neq 0$ then $a_i = 0$. It follows that $\mathbf{z}$ is generated by the self-orthogonal vectors of the basis, which, being independent, therefore form a basis of $\mathcal{C} \cap \mathcal{C}^\perp$. ☐

Proposition 4.42. *Let $\mathcal{C}$ be a subspace of $\mathbb{F}_3^E$. Then*

$$\sum_{\mathbf{z} \in \mathcal{C}} \omega^{|\mathrm{supp}(\mathbf{z})|} = (-1)^{d + d_1} (i\sqrt{3})^{d + d_0},$$

where $d = \dim \mathcal{C}$, $d_0 = \dim(\mathcal{C} \cap \mathcal{C}^\perp)$, and d_1 is the number of basis vectors of support size congruent to 1 modulo 3 in any orthogonal basis of $\mathcal{C}$.

Proof. Observe that for $\mathbf{z} \in \mathbb{Z}_3^E$ we have $|\mathrm{supp}(\mathbf{z})| \equiv \mathbf{z}^\top \mathbf{z} \pmod 3$. It follows that $\omega^{|\mathrm{supp}(\mathbf{z})|} = \omega^{\mathbf{z}^\top \mathbf{z}}$.

By Lemma 4.40 there is an orthogonal basis $\{\mathbf{z}_1, \ldots, \mathbf{z}_d\}$ of $\mathcal{C}$. In particular, the inner product of $\mathbf{z} = \sum_{1\le j\le d} a_j\mathbf{z}_j$ with itself is equal to $\sum_{1\le j\le d} a_j^2\mathbf{z}_j^\top\mathbf{z}_j$. So we find that

$$\begin{aligned}\sum_{\mathbf{z}\in\mathcal{C}}\omega^{\mathbf{z}^\top\mathbf{z}} &= \sum_{(a_1,\ldots,a_d)\in\mathbb{Z}_3^d}\omega^{\sum_{1\le j\le d}a_j^2\mathbf{z}_j^\top\mathbf{z}_j}\\ &= \sum_{(a_1,\ldots,a_d)\in\mathbb{Z}_3^d}\prod_{1\le j\le d}\omega^{a_j^2\mathbf{z}_j^\top\mathbf{z}_j}\\ &= \prod_{1\le j\le d}\sum_{a_j\in\mathbb{Z}_3}\omega^{a_j^2\mathbf{z}_j^\top\mathbf{z}_j}\\ &= \prod_{1\le j\le d}(1+2\omega^{\mathbf{z}_j^\top\mathbf{z}_j})\\ &= 3^{d_0}(1+2\omega)^{d_1}(1+2\omega^2)^{d-d_0-d_1},\end{aligned}$$

where d_0 (resp. d_1) is the number of vectors $\mathbf{z}_j, 1\le j\le d$, such that $\mathbf{z}_j^\top\mathbf{z}_j = 0$ (resp. $=1$). With $1+2\omega = i\sqrt{3}$, $1+2\omega^2 = -i\sqrt{3}$, and $d_0 = \dim(\mathcal{C}\cap\mathcal{C}^\perp)$ by Lemma 4.41, the statement of the proposition now follows. $\square$

As Gioan and Las Vergnas [17] observe in their Corollary 2, it is not obvious that the parity of the number of vectors in an orthogonal basis for $\mathcal{C}$ with support size congruent to 1 modulo 3 is independent of the choice of basis, a fact implied by Proposition 4.42.

We reach another polynomial time computable evaluation of the Tutte polynomial (bases for finite-dimensional vector spaces being easy to find by Gaussian elimination, and Lemma 4.40 providing a polynomial time algorithm for constructing an orthogonal basis):

Theorem 4.43. *Let $G = (V, E)$ be a graph and $\omega = e^{2\pi i/3}$. We have*

$$T(G; \omega, \omega^2) = (-1)^{d_2}\omega^{|E|+d}(i\sqrt{3})^{d_0},$$

where d_0 is the dimension of the space of $\mathbb{Z}_3$-tension-flows of G, d the dimension of the space of $\mathbb{Z}_3$-flows, and d_2 is the number of vectors with support size congruent to 2 modulo 3 in any orthogonal basis for the space of $\mathbb{Z}_3$-flows.

Proof. Setting $k = 3$ and $x = \omega^2 = \omega^{-1}$ in equation (18) we have

$$\sum_{\mathbf{z}\in\mathcal{C}}\omega^{-|E|+|\mathrm{supp}(\mathbf{z})|} = (\omega^2-1)^d T(G; \omega^2, \omega),$$

where $d = \dim\mathcal{C} = n(G)$ is the dimension of the space of $\mathbb{Z}_3$-flows. Then by Proposition 4.42 and $\omega^2 - 1 = i\sqrt{3}\omega$ we obtain

$$\omega^{-|E|}(-1)^{d+d_1}(i\sqrt{3})^{d+d_0} = (i\sqrt{3}\omega)^d T(G; \omega^2, \omega).$$

Since $T(G; \omega^2, \omega)$ is the complex conjugate of $T(G; \omega, \omega^2)$ the result follows. $\square$

In Section 4.4 we saw that $T(G; -1, -1) = (-1)^{|E(G)|}(-2)^{b(G)}$, where $b(G)$ is the bicycle dimension of G, i.e., the dimension of the the subspace of $\mathbb{Z}_2$-tension-flows. The point $(-1, -1)$ lies on the hyperbola $(x-1)(y-1) = 4$, so that by identity (18)

$$T(G; -1, -1) = (-2)^{-n(G)}\sum_{\mathbb{Z}_2\,\times\,\mathbb{Z}_2\text{-flows }\mathbf{z}}(-1)^{|E|-|\mathrm{supp}(\mathbf{z})|}.$$

This might lead one to expect rather an expression for $T(G;-1,-1)$ in terms of the space of $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows in $\mathbb{F}_4^E$. Indeed, the dimension of the space of $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows is equal to the bicycle dimension $b(G)$. A $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flow decomposes by projection into a pair of $\mathbb{Z}_2$-tension-flows, and conversely such a pair of $\mathbb{Z}_2$-tension-flows can be pieced together to make a $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flow. Hence there are precisely $(2^{b(G)})^2$ vectors that are $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows, i.e., they comprise a space of dimension $b(G)$ over $\mathbb{F}_4$. Hence we could also have written that $T(G;-1,-1)=(-1)^{|E|}(-2)^{d_0}$, where d_0 is the dimension of the space of $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows.

Exercise 4.44. *Are there in general as many $\mathbb{Z}_4$-tension-flows as $\mathbb{Z}_2\times\mathbb{Z}_2$-tension-flows?*

Vertigan proved that the Tutte polynomial evaluated at the point $(i,-i)$ on the hyperbola $(x-1)(y-1)=2$ has the following interpretation:

Theorem 4.45 ([50])**.** *Let G be a graph with bicycle dimension $b(G)$. Then*

$$|T(G;i,-i)|=\begin{cases}\sqrt{2}^{b(G)} & \text{if every bicycle has size a multiple of 4,}\\ 0 & \text{otherwise.}\end{cases}$$

For example, $T(C_4;i,-i)=i^3+i^2+i-i=-i-1=-\sqrt{2}\frac{1+i}{\sqrt{2}}$, where $\frac{1+i}{\sqrt{2}}$ is a primitive eighth root of unity. Recall also that every bicycle has even size, so that the bicycles of size a multiple of 4 either comprise all bicycles, or exactly half of them. Theorem 4.45 implies a polynomial time algorithm for evaluating $T(G;i,-i)$.

5 Computational complexity

We have seen that the Tutte polynomial can be computed in polynomial time at some particular points. Specifically, these points are: $(0,0)$ (whether there are any edges), $(1,1)$ (number of spanning trees), $(2,2)$ (number of subgraphs), $(-1,0)$ (whether bipartite or not), $(0,-1)$ (whether Eulerian or not), $(-1,-1)$ (up to easily determined sign equal to number of bicycles), and also in the last section interpretations for evaluations at $(e^{2\pi i/3},e^{-2\pi i/3})$ and $(i,-i)$, the former involving the dimension the space spanned by vectors that are simultaneously $\mathbb{Z}_3$-flows and $\mathbb{Z}_3$-tensions.

Recall also that $T(G;x,y)=(x-1)^{r(G)}y^{|E(G)|}$ when $(x-1)(y-1)=1$, so that the Tutte polynomial is also polynomial time computable at points on this hyperbola (the points $(0,0)$ and $(2,2)$ were already mentioned in the previous paragraph).

Theorem 5.1 below says that we have in fact now encountered all such "easy points".

A computational (enumeration) problem can be regarded as a function mapping inputs to solutions (graphs to the number of their proper vertex 3-colourings, for example). A problem is *polynomial time computable* if there is an algorithm which computes the output in length of time (number of steps) bounded by a polynomial in the size of the problem instance. The class of such problems is denoted by P. If A and B are two problems, we say that A is *polynomial time reducible* to B, written $A\propto B$, if it is possible with the help of a subroutine for problem B to solve problem A is polynomial time.

The class $\#\mathsf{P}$ can be roughly described as the class of all enumeration problems in which the structures being counted can be recognized in polynomial time (i.e., instances of an NP problem). For example, counting Hamiltonian paths in a graph is in $\#\mathsf{P}$ because it is easy to check whether a given set of edges is a Hamitonian path.

The class $\#\mathsf{P}$ has a class of "hardest" problems called the $\#\mathsf{P}$-complete problems. A problem A belonging to $\#\mathsf{P}$ is $\#\mathsf{P}$-complete if for any other problem B in $\#\mathsf{P}$ we have $B \propto A$. A prototypical example of a $\#\mathsf{P}$-complete problem is #Sat, the problem of counting the number of satisfying assignments of a Boolean function. Many of the thousands of problems known to be $\#\mathsf{P}$ complete have been shown to be so by reduction to #Sat. Counting Hamiltonian paths is an example of a $\#\mathsf{P}$-complete problem (even when restricted to planar graphs with maximum degree 3).

A problem is $\#\mathsf{P}$-hard if any problem in $\#\mathsf{P}$ is polynomial time reducible to it. In other words, A is $\#\mathsf{P}$-hard if the existence of a polynomial time algorithm for A would imply the existence of a polynomial time algorithm for any problem in $\#\mathsf{P}$. (A $\#\mathsf{P}$-hard problem is $\#\mathsf{P}$-complete if it belongs to the class $\#\mathsf{P}$ itself.)

We have found that many evaluations of the Tutte polynomial count structures associated with a graph. Sometimes though it is not apparent what an evaluation of the Tutte polynomial at a particular point (a, b) might count. However, we can still speak of whether the problem of computing $T(G; a, b)$ can be done in polynomial time or if it is a $\#\mathsf{P}$-hard problem (being able to evaluate it for any graph in polynomial time would imply that every problem in $\#\mathsf{P}$ could be computed in polynomial time).

Theorem 5.1 ([27])**.** *Evaluating the Tutte polynomial of a graph at a particular point of the complex plane is $\#\mathsf{P}$-hard except when either*

(i) the point lies on the hyperbola $(x-1)(y-1) = 1$,

(ii) the point is one of the special points $(1,1)$, $(-1,0)$, $(0,-1)$, $(-1,-1)$, $(i,-i)$, $(-i,i)$, $(e^{2\pi i/3}, e^{-2\pi i/3})$, $(e^{-2\pi i/3}, e^{2\pi i/3})$.

In the special cases (i) and (ii) evaluation can be carried out in polynomial time.

In [48] Vertigan and Welsh show that the same statement in Theorem 5.1 holds even when restricting the problem to computing the Tutte polynomial for bipartite graphs.

Around the same time as [48], but only much later published, Vertigan showed that restricting the problem of evaluating the Tutte polynomial to planar graphs only yields extra "easy points" on the hyperbola $(x-1)(y-1) = 2$ (corresponding to the partition function of the Ising model, which in the planar case is polynomial time computable due to Kasteleyn's expression for the partition function of the Ising model as the Pfaffian of an associated matrix).

Theorem 5.2 ([49])**.** *The problem of computing the Tutte polynomial of a planar graph at a particular point of the complex plane is $\#\mathsf{P}$-hard except when either*

(i) the point lies on one the hyperbolae $(x-1)(y-1) = 1$ or $(x-1)(y-1) = 2$,

(ii) the point is one of the special points $(1,1)$, $(-1,0)$, $(0,-1)$, $(-1,-1)$, $(e^{2\pi i/3}, e^{-2\pi i/3})$, $(e^{-2\pi i/3}, e^{2\pi i/3})$.

In the special cases (i) and (ii) evaluation can be carried out in polynomial time.

See e.g. [52] for a more detailed account of the complexity of counting problems, with special emphasis on those related to the Tutte polynomial.

6 The Tutte polynomial in statistical physics

For more on physical models related to the Tutte polynomial see [52, Chapter 4], [30] and [41]. For combinatorics associated with the Ising model see [33] and also the book [34]. Sokal's papers [44] and [45] give a lucid explanation of how combinatorial properties of the partition function of the general Potts model correspond to physical properties of a system. For more about exactly solved models in statistical physics see [2].

6.1 The Ising model

In the general Ising model on a graph $G = (V, E)$ each vertex i of G is assigned a *spin* σ_i, which is either $+1$ ("up") or -1 ("down"). An assignment of spins to all the vertices of G is called a *configuration* or *state* and denoted by σ.

Each edge $e = ij$ of G has an *interaction energy* J_{ij} which is constant on the edge, but may vary from edge to edge.

For each state σ the Hamiltonian $H(\sigma)$ is defined by

$$H(\sigma) = -\sum_{ij\in E} J_{ij}\sigma_i\sigma_j - \sum_{i\in V} M\sigma_i,$$

where M represents the energy of the external field.

The Hamiltonian $H(\sigma)$ measures the energy of the state σ. In a ferromagnet the J_{ij} are positive, which has the consequence that a configuration of spins in which adjacent vertices have parallel spins ($\sigma_i = \sigma_j$ for $ij \in E$) has a lower energy than a non-magnetized state in which spins are arbitrary. The external field has the effect of aligning spins with the direction of the field, thus again favouring states of low energy.

The *partition function* $Z = Z(G; \beta, J, M)$ is defined by

$$Z(G) = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where the sum is over all $2^{|V|}$ possible spin configurations and $\beta = 1/kT$ is a parameter determined by the absolute temperature T and where k is Boltzmann's constant. The probability of finding the system in a state σ is given by

$$\Pr(\sigma) = e^{-\beta H(\sigma)}/Z(G).$$

This is the probability distribution on states σ which has maxiumum entropy for a given mean value $-\frac{\partial}{\partial\beta}\log Z(G)$ of the energy $H(\sigma)$. See [23] and [24] for more on information theory in statistical physics. A high temperature gives a low value of β and the probability distribution of states becomes more flat. On the other hand, a low temperature gives high β and correspondingly greater probability to low energy states.

The entropy of a finite probability distribution $(p_1, \ldots p_N)$ is defined by

$$h(p_1, \ldots, p_N) = -\sum_k p_k \log_2 p_k,$$

and is a measure of uncertainty in the system whose states follow the given distribution. The entropy of the Ising model system is

$$h(G; \beta) = -\sum_{\sigma} \Pr(\sigma)\log_2 \Pr(\sigma),$$

which gives

$$h(G;\beta) = \left[\frac{1}{Z(G)}(\beta \log_2 e) \sum_\sigma H(\sigma) e^{-\beta H(\sigma)}\right] + \log_2 Z(G).$$

Seeing that

$$\frac{\partial}{\partial \beta} \log Z(G) = \frac{1}{Z(G)} \frac{\partial Z(G)}{\partial \beta} = -\sum_\sigma \frac{H(\sigma)}{Z(G)} e^{-\beta H(\sigma)},$$

we have

$$h(G;\beta) = -(\beta \log_2 e) \frac{\partial}{\partial \beta} \log Z(G) + \log_2 Z(G).$$

The quantity $-\frac{\partial}{\partial \beta} \log Z(G)$ is called the *internal energy* and the quantity $\log Z(G) = \log_2 Z(G)/\log_2 e$ is the *free energy*.

Consider some countably infinite graph such as the two-dimensional square lattice (vertices $\mathbb{Z}^2$, with vertex (a,b) adjacent to vertices $(a\pm 1, b)$ and $(a, b\pm 1)$) and an increasing sequence of finite subgraphs $G_n = (V_n, E_n)$. Then, under reasonable hypotheses on the G_n, it can be shown that the *(limiting) free energy per lattice site*

$$\lim_{n\to\infty} \frac{\log Z(G_n)}{|V_n|}$$

exists for non-degenerate physical values of the parameters β, J, M of $Z(G)$.

Complex singularities of $\log Z(G_n)$ (i.e., zeroes of $Z(G_n)$) may approach the real axis in the limit $n \to \infty$, and in this case the points of physical *phase transitions* are precisely the real limit points of such complex zeroes. In the ferromagnetic Ising model (positive interaction energies J_{ij}), a cooling slab of iron becomes magnetized at the *critical temperature* that gives a phase transition.

In particular, the main problem of the Ising model on the two-dimensional lattice is to determine

$$\lim_{n\to\infty} \frac{\log Z(L_{n,n})}{n^2}$$

where $L_{n,n}$ is the $n \times n$ grid. (In practice, in order to facilitate anlysis $L_{n,n}$ is replaced by the $n \times n$ toroidal grid.)

6.1.1 Constant interaction energies, no external field

Assume that $M = 0$, so that there is no external field, and that $J_{ij} = J$ is constant over all edges of G.

The partition function is now

$$Z(G) = Z(G;\beta, J) = \sum_\sigma e^{-\beta H(\sigma)},$$

where

$$H(\sigma) = -\sum_{ij\in E} J\sigma_i\sigma_j.$$

Theorem 6.1. *The partition function for the Ising model on $G = (V,E)$ when there is constant edge interaction J and no external field is given by*

$$Z(G) = 2^{|V|} e^{-\beta J n(G)} (\sinh \beta J)^{r(G)} T(G; \coth \beta J, e^{2\beta J}),$$

where $\beta = 1/kT$.

Proof. Consider a configuration σ as a vertex 2-colouring of G with colours -1 and $+1$. The contribution to $e^{-\beta H(\sigma)}$ of the edge ij when $\sigma_i = \sigma_j$ is equal to $e^{\beta J}$ and when $\sigma_i \neq \sigma_j$ its contribution is $e^{-\beta J}$. In other words we have, in terms of the monochrome polynomial $B(G; 2, y)$ with $y = e^{2\beta J}$,

$$\begin{aligned}
Z(G) &= \sum_{\sigma} e^{2\beta J \#\{ij \in E : \sigma_i = \sigma_j\} - \beta J|E|} \\
&= e^{-\beta J|E|} B(G; 2, e^{2\beta J}) \\
&= 2^{c(G)} e^{-\beta J|E|} (e^{2\beta J} - 1)^{r(G)} T(G; \frac{e^{2\beta J} + 1}{e^{2\beta J} - 1}, e^{2\beta J}).
\end{aligned}$$

This can be written in terms of hyperbolic functions as given in the theorem statement. □

Using flow-tension duality (19), we can also express the partition function for the Ising model in terms of the weight enumerator for 2-flows (18) (Eulerian subgraphs):

$$\begin{aligned}
Z(G) &= e^{-\beta J|E|} B(G; 2, e^{2\beta J}) \\
&= 2^{|V|-|E|} (e^{\beta J} - e^{-\beta J})^{|E|} C(G; 2, \frac{e^{2\beta J} + 1}{e^{2\beta J} - 1}) \\
&= 2^{|V|} (\sinh \beta J)^{|E|} C(G; 2, \coth \beta J).
\end{aligned}$$

Consequently we have the following:

Theorem 6.2 (Van der Waerden, 1941)**.**

$$Z(G) = 2^{|V|} (\cosh \beta J)^{|E|} \widetilde{C}(G; \tanh \beta J),$$

where

$$\widetilde{C}(G; x) = \sum_{\substack{A \subseteq E \\ (V,A) \text{ Eulerian}}} x^{|A|} = x^{|E|} C(G; 2, x^{-1}).$$

The Eulerian subgraph expansion of the partition function of the Ising model of Theorem 6.2 is the starting point for reducing the Ising model problem for square lattices to a dimer (matching) problem, and thence via Pfaffian orientations to Onsager's solution in 1944 of the problem of finding $\lim_{n \to \infty} \frac{\log Z(L_{n,n})}{n^2}$. In particular it enabled the critical temperature T_c to be found for the two-dimensional lattice.

A non-rigorous argument for finding that the critical temperature is given by $kT_c/J = \frac{2}{\log(1+\sqrt{2})}$ had already been known before Onsager's solution to the two-dimensional Ising problem. One approach is to use the self-duality of the infinite plane lattice, and "approximate self-duality" of its finite lattice subgraphs $L_{n,n}$ for large n.

Let $\gamma = \beta J$. We have

$$\begin{aligned}
Z(G) &= e^{-\gamma|E|} B(G; 2, e^{2\gamma}) \\
&= 2^{|V|} (\cosh \gamma)^{|E|} \widetilde{C}(G; \tanh \gamma).
\end{aligned}$$

When $G = (V, E)$ is a connected planar graph with dual G^*, by flow-tension duality we have

$$B(G^*; 2, e^{2\gamma}) = 2C(G; 2, e^{2\gamma}) = 2e^{2\gamma|E|} \widetilde{C}(G; e^{-2\gamma}).$$

Assume now that $G \cong G^*$ is self-dual (like the infinite square lattice). We then have the *low temperature expansion*

$$Z(G;\gamma) = 2e^{\gamma|E|}\widetilde{C}(G;e^{-2\gamma}), \tag{20}$$

where as $T \to 0$ so $\gamma \to \infty$, and the partition function is dominated by terms $e^{\gamma(|E|-2|A|)}$ for which $|A|$ is small. We also have the *high temperature expansion*,

$$Z(G;\gamma) = 2^{|V|}(\cosh\gamma)^{|E|}\widetilde{C}(G;\tanh\gamma), \tag{21}$$

where as $T \to \infty$ so $\gamma \to 0, \tanh\gamma \to 0$, and the partition function is dominated by terms $(\tanh\gamma)^{|A|}$ for which $|A|$ is small.

By equations (20) and (21), the free energy per vertex is given by

$$\frac{\log Z(G;\gamma)}{|V|} = \frac{\log 2}{|V|} + \frac{\gamma|E|}{|V|} + \frac{\log \widetilde{C}(G;e^{-2\gamma})}{|V|}$$

and also by

$$\frac{\log Z(G;\gamma)}{|V|} = \frac{|V|\log 2}{|V|} + \frac{|E|\cosh\gamma}{|V|} + \frac{\log \widetilde{C}(G;\tanh\gamma)}{|V|}.$$

Take $G_n = (V_n, E_n)$ to be a 4-regular planar graph for which G_n converges as $n \to \infty$ to the infinite plane lattice. We have $2|V_n| = |E_n|$, and, letting $|V_n| \to \infty$, (it is here we use the "approximate self-duality" of the graphs G_n)

$$2\gamma + \lim_{n\to\infty} \frac{\log \widetilde{C}(G_n;e^{-2\gamma})}{|V_n|} = \log 2 + 2\cosh\gamma + \lim_{n\to\infty} \frac{\log \widetilde{C}(G_n;\tanh\gamma)}{|V_n|}.$$

If we choose γ^* such that $\tanh\gamma^* = e^{-2\gamma}$ and let

$$F(\gamma) = \lim_{n\to\infty} \frac{\log \widetilde{C}(G_n;e^{-2\gamma})}{|V_n|},$$

then we have

$$2\gamma + F(\gamma) = \log(2\cosh^2\gamma) + F(\gamma^*).$$

The left-hand side comes from the low temperature expansion, the right-hand side from the high temperature expansion (when γ is small γ^* is large). Assuming that there is only one critical point γ_c (corresponding to critical temperature $T_c = \frac{J}{k\gamma_c}$), then $\gamma_c^* = \gamma_c$ and we obtain

$$2\gamma_c = \log(2\cosh^2\gamma_c),$$

from which we find

$$\gamma_c = \frac{\log(1+\sqrt{2})}{2}.$$

6.2 Ice-type model (Six-vertex model)

Square ice consists of an $n \times n$ lattice arrangement of oxygen atoms. Between any two adjacent O-atoms lies one hydrogen atom, and there are also H-atoms at the left and right boundaries. The problem is to count all possible configurations in which evey O-atom is attached to exactly two of its surrounding H-atoms, forming H_2O.

There is a bijection between $n \times n$ ice configurations and Eulerian orientations on the lattice graph of O-atoms, with boundary conditions. Let u and v be two adjacent O-atoms. Orient the edge $u \longrightarrow v$ if the H-atom between u and v is attached to v. On the left and right boundaries all edges are incoming (each H-atom on the boundary is attached to an O-atom horizontally). On the top and bottom boundaries all edges are outgoing.

In this way we get an Eulerian orientation of the $n \times n$ lattice graph with hanging boundary edges (each missing one endpoint).

The number of ice configurations is the number of Eulerian orientations of the $n \times n$ lattice graph with boundary conditions (incoming edges left and right, outgoing edges top and bottom). Each O-atom has six possible attachments to neighbouring H-atoms, corresponding to the six possible orientations at a vertex of degree 4 with two incoming and two outgoing edges. (This gives the name "Six-vertex model".)

The $n \times n$ lattice graph of O-atoms with directed edges added as described gives an $(n+1) \times (n+1)$ array of square cells, where each O-vertex is incident with four cells. The cells can be $\mathbb{Z}_3$-coloured by the following rule. Colour the top left corner 0. Suppose a and b are neighbouring cells such that the edge that separates them has orientation having a to the left and b to the right, and that a and b have colours $c(a)$ and $c(b)$ respectively. Then $c(b) = c(a) + 1$. In other words add one modulo 3 going from left to right across a directed edge. The boundary colours appear in sequence $0, 1, 2, 0, \ldots$, with the bottom right corner coloured 0 like the top left. (The sequence along the top is the mirror image of that along the bottom, and likewise for left and right boundaries.)

This gives a bijection between $n \times n$ ice configurations and proper $\mathbb{Z}_3$-colourings of the $(n+1) \times (n+1)$-array of cells, observing the boundary conditions.

An alternative way to see this 3-colouring procedure is to first add edges to the $n \times n$ lattice graph $L_{n,n}$ to make it a 4-regular graph as follows. Given $L_{n,n}$ on vertex set $[n] \times [n]$, add edges between $(i, 1)$ and $(1, i)$ for each $i \in [n]$ and edges between (i, n) and (n, i) for each $i \in [n]$. This yields a 4-regular planar graph $\widetilde{L}_{n,n}$ (with loops at the two corners $(1, 1)$ and (n, n)). An Eulerian orientation of $\widetilde{L}_{n,n}$ is obtained by the same rule of directing O-atom u towards O-atom v when v is attached to the H-atom between u and v, the orientation of edges joining boundary O-atoms being determined by always directing edge into those vertices on the left or right boundaries. By tension-flow duality, each nowhere-zero $\mathbb{Z}_3$-flow (Eulerian orientation) of $\widetilde{L}_{n,n}$ corresponds to a nowhere-zero $\mathbb{Z}_3$ tension of the dual graph $\widetilde{L}^*_{n,n}$, i.e. to three proper $\mathbb{Z}_3$-colourings of the faces of $\widetilde{L}_{n,n}$. Fixing the colour of either of the loop faces to be 0, it is easy to see that this corresponds to the cell-colouring described above.

This 3-coloured version of the square ice problem is the starting point for the proof of the remarkable formula obtained by Zeilberger and Kuperberg in 1996: the number of $n \times n$ ice configurations is equal to

$$\frac{(3n-2)!(3n-5)! \cdots 4!1!}{(2n-1)!(2n-2)! \cdots (n+1)!n!}.$$

See [1, Chapter 10] and [7].

In the general case, an ice model concerns the number of ways of orienting a 4-regular graph G such that each vertex has 2 incoming edges and 2 outgoing edges, i.e., an Eulerian orientation of G. Let $\mathrm{ice}(G)$ denote this number. When each state is equiprobable and of the same energy (as for square ice), the partition function is given by $Z(G) = \mathrm{ice}(G)$ and the free energy is $\log Z(G)$.

In Proposition 4.13 we saw that Eulerian orientations of a 4-regular graph correspond to nowhere-zero $\mathbb{Z}_3$-flows of G, so that we have

$$\mathrm{ice}(G) = (-1)^{n(G)} T(G; 0, -2).$$

Although finding an Eulerian orientation can be done polynomial time, in general computing the number of them is #P-complete, as proved by Mihail and Winkler [38]. (Note that Theorem 5.1 says that finding the number of nowhere-zero $\mathbb{Z}_3$-flows, $(-1)^{n(G)}T(G; 0, -2)$, is also a #P-complete problem; this number coincides with the number of Eulerian orientations on the class of 4-regular graphs.)

Proposition 6.3. *Let $G = (V, E)$ be a 4-regular graph. Then* $\mathrm{ice}(G) \geq \left(\frac{3}{2}\right)^{|V|}$.

Proof. [sketch – argument was given in full in lecture on 13.12.10]

Use induction on the number of vertices of G. The case of a single vertex with two loops has $\mathrm{ice}(G) = 4 \geq \frac{3}{2}$.

For a graph on n vertices, choose one, say v, and partition Eulerian orientations of G according to which of the six possible configurations is at v. Fix an Eulerian orientation of G. Let a, b, c, d be the neighbours of v and suppose that $a \longrightarrow v$, $b \longrightarrow v$, $v \longrightarrow c$, $v \longrightarrow d$.

Define a 2-in 2-out digraph G_1 on vertex set $V \setminus \{v\}$ as follows. Take the same edge orientations as G for edges not incident with v, together with directed edges $a \longrightarrow c$, $b \longrightarrow d$ to replace the four edges of G incident with v. Similarly, define the 2-in 2-out digraph G_2 by in a similar way except taking directed edges $a \longrightarrow d$ and $b \longrightarrow c$.

Depending on which of the six possible configurations of directed edges is at v, the digraphs G_1 and G_2 are Eulerian orientations of two of three possible 4-regular graphs $G_\alpha, G_\beta, G_\gamma$.

After some case analysis we then find that

$$\mathrm{ice}(G_\alpha) + \mathrm{ice}(G_\beta) + \mathrm{ice}(G_\gamma) \leq 2\,\mathrm{ice}(G),$$

and by induction hypothesis

$$3 \cdot \left(\frac{3}{2}\right)^{n-1} \leq 2\,\mathrm{ice}(G),$$

yielding the desired lower bound. □

In the square ice model we take $G \cong \widetilde{L}_{n,n}$ the $n \times n$ grid with edges added between $(i, 1)$ and $(1, i)$ and edges between (i, n) and (n, i), for each $i \in [n]$.

Lieb proved in 1967 that for the square lattice

$$\lim_{n\to\infty} \mathrm{ice}(\widetilde{L}_{n,n})^{\frac{1}{n^2}} = \left(\frac{4}{3}\right)^{\frac{3}{2}} \approx 1.5396.$$

This is quite close to the lower bound of $\frac{3}{2}$ given by Proposition 6.3.

6.3 The Potts model

The q-state Potts model on a graph $G = (V, E)$ is a generalization of the Ising model in which there are q possible states at a vertex rather than the two up/down states. In this model introduced by Askin and Teller (1943) and Potts (1952) the energy between two adjacent spins at vertices i and j is taken to be zero if the spins are the same and equal to a constant J_{ij} if they are different. For a state σ the Hamiltonian is defined by

$$H(\sigma) = \sum_{ij \in E} J_{ij}(1 - \delta(\sigma_i, \sigma_j)),$$

where δ is the Kronecker delta function ($\delta(a,b) = 1$ if $a = b$ and $\delta(a,b) = 0$ if $a \neq b$). We shall assume there is no external magnetic field. The Hamiltonian $H(\sigma)$ represents the energy of the state σ. The partition function of the q-state Potts model is defined by

$$Z(G) = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where the sum is over all $q^{|V|}$ possible states σ and β is the inverse temperature $\beta = \frac{1}{kT}$ as for the Ising model.

Just as for the Ising model, we have

$$\Pr(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z(G)},$$

the Boltzmann maximum entropy distribution on the state space subject to a given expected value of $H(\sigma)$. (This expected value is the internal energy of the system, which is constant when the system is isolated/ in equilibrium with its environment. This is the First Law of Thermodynamics, expressing the principle of conservation of energy.)

If we replace J_{ij} by $-2J_{ij}$ then the partition function of the 2-state Potts model is the same as that of the Ising model scaled by $e^{-\beta \sum_{ij \in E} J_{ij}}$.

Returning to the q-state Potts model, if $J_{ij} = J$ is constant over all edges and we write $K = \beta J$ then the partition function can be written in the following ways:

$$\begin{aligned}
Z(G) &= \sum_{\sigma \in [q]^V} e^{-K(|E| - \#\{ij \in E : \sigma_i = \sigma_j\})} \\
&= e^{-K|E|} B(G; q, e^K) \\
&= q^{|V|-|E|}(1 - e^{-K})^{|E|} C(G; q, \frac{e^K - 1 + q}{e^K - 1}) \\
&= q^{c(G)}(e^K - 1)^{r(G)} e^{-K|E|} T(G; \frac{e^K - 1 + q}{e^K - 1}, e^K).
\end{aligned}$$

The point $(\frac{e^K-1+q}{e^K-1}, e^K)$ lies on the hyperbola $(x-1)(y-1) = q$.

Here is a summary of correspondences between the Potts model and the Tutte plane (taken from [51]):

Potts model on G	Tutte polynomial $T(G; x, y)$
Ferromagnetism	Positive ($x, y > 1$) branch of $(x-1)(y-1) = q$
Antiferromagnetism	Negative ($x < 0$) branch of $(x-1)(y-1) = q$ with $y > 0$
High temperature	Asymptote of $(x-1)(y-1) = q$ to $y = 1$
Low temp. ferromagnetic	Positive branch of $(x-1)(y-1) = q$ asymptotic to $x = 1$
Zero temp. antiferromagnetic	Proper vertex q-colourings, $x = 1 - q, y = 0$.

6.4 The Fortuin-Kasteleyn random cluster model

The random cluster model on a connected graph $G = (V, E)$ with parameters p and q is a probability space on all spanning subgraphs of G. The probability measure of a subgraph $A \subseteq E$ is

$$\mu(A) = \frac{1}{Z(G)} p^{|A|}(1-p)^{|E\setminus A|} q^{c(A)},$$

where as usual $c(A)$ denotes the number of connected components of the subgraph (V, A), and $Z(G)$ is the normalizing constant

$$Z(G) = \sum_{A\subseteq E} p^{|A|}(1-p)^{|E\setminus A|} q^{c(A)}.$$

When $q = 1$ this is the *bond percolation* model on G, where an edge is open with probability p and otherwise closed. This model is used for such processes as molecules penetrating a porous solid, diffusion, and the spread of infection through a community (passage/contagion is possible along open edges).

Letting $q \to 0$, a subgraph has non-zero probability if and only if it is connected and in this case the partition function is the reliability polynomial:

$$\begin{aligned} Z(G) &= \sum_{A\subseteq E} p^{|A|}(1-p)^{|E\setminus A|} \\ &= (1-p)^{|E|-|V|+1} p^{|V|-1} T(G; 1, \frac{1}{1-p}), \end{aligned}$$

(see Proposition 3.10).

When q is a positive integer the random cluster model is equivalent to the q-state Potts model with $p = 1 - e^{-K}$. Using the subgraph expansion of the Tutte polynomial we have the following:

Proposition 6.4. *The partition function of the random cluster model on a connected graph $G = (V, E)$ with parameters $0 \le p \le 1$ and $q > 0$ is given by*

$$Z(G) = q(1-p)^{|E|-|V|+1} p^{|V|-1} T(G; 1 + \frac{(1-p)q}{p}, \frac{1}{1-p}),$$

and the probability measure of the subgraph A is given by

$$\mu(A) = \frac{\left(\frac{p}{1-p}\right)^{|A|} q^{c(A)-1}}{\left(\frac{p}{(1-p)}\right)^{|V|-1} T(G; \frac{p+q-pq}{p}, \frac{1}{1-p})}.$$

When $q > 1$ there is a bias towards edges joining vertices in an existing component than edges uniting two old components, since a larger number of components are favoured. More precisely, given $B \subseteq E$ and $e \in E \setminus B$, under the probability distribution μ we have

$$\begin{aligned} \Pr(e \in A \,|\, A \setminus \{e\} = B) &= \frac{\Pr(A = B \cup \{e\})}{\Pr(A - \{e\} = B)} = \frac{\mu(B \cup \{e\})}{\mu(B \cup \{e\}) + \mu(B)} \\ &= \begin{cases} p & \text{if } c(B \cup \{e\}) = c(B), \\ \frac{p}{p+q(1-p)} & \text{if } c(B \cup \{e\}) = c(B) - 1, \end{cases} \end{aligned}$$

where, for $0 < p < 1$,

$$\frac{p}{p+q(1-p)} \begin{cases} < p & \text{if } q > 1 \\ > p & \text{if } 0 < q < 1. \end{cases}$$

Percolation in the random cluster model (the existence of an infinite component of open edges) is intimately related to *two-point correlation* (long-distance correlation between vertex colours) in the q-state Potts model. Given fixed vertices i and j, in the Ising model the two-point correlation between i and j is defined to be the expected value of $\sigma_i\sigma_j$ over all states σ. For the Potts model the two-point correlation is the expected value of $\delta(\sigma_i, \sigma_j)$, i.e., the probability that σ_i equals σ_j.

A key result of Fortuin and Kasteleyn (1969) is the following (see e.g. [21, Theorem 2.1]):

Theorem 6.5. *For any pair of vertices i and j and positive integer q, the probability that σ_i equals σ_j in the q-state Potts model is given by*

$$\frac{1}{q} + (1 - \frac{1}{q})\mu\{i \rightsquigarrow j\},$$

where μ is the random cluster probability measure on G obtained by taking $p = 1 - e^{-K}$ and $\{i \rightsquigarrow j\}$ is the event that there is an open path from i to j, i.e.,

$$\{i \rightsquigarrow j\} = \bigcup \{A \subseteq E : i \text{ and } j \text{ belong to the same component of } (V, A)\}.$$

The expression on the right-hand side in Theorem 6.5 can be regarded as being made up of two parts. The first term $1/q$ is the probability that under a uniformly random colouring of the vertices of G the vertices i and j have the same colour. The second term measures the probability of long-range interaction. So Theorem 6.5 expresses an equivalence between long-range spin correlations and percolatory behaviour.

Phase transition (in the infinite system) occurs at the onset of an infinite cluster (connected component) in the random cluster model and corresponds to spins on the vertices of the Potts model having long-range two-point correlation.

See [52, Chapter 4] for further discussion of percolation in the random cluster model, as well as the detailed account of [22] from the point of view of probability theory.

6.5 Graph homomorphisms

Many generalizations of the Tutte polynomial have been studied that have been motivated by applications in statistical physics (see e.g. [45] for the multivariate Tutte polynomial, equivalent to the partition function of the general Potts model where edge interactions vary from edge to edge), and by knot theory (see e.g. [40] for the U-polynomial), as well as the V-functions studied by Tutte himself, these being the most general multivariate polynomials which satisfy a deletion-contraction recurrence whose parameters may depend on which particular edge is being deleted/contracted.

Another perspective is to regard the chromatic polynomial, and more generally, the partition function of the q-state Potts model on a graph G, as arising from counting homomorphisms from G to a graph H (possibly with weights on its edges). For example, $P(G; k)$ is equal to the number of homomorphisms from G to K_k (think of the vertices of K_k as being colours). More generally, the monochrome polynomial $B(G; k, y)$ is the number of homomorphisms from G to the complete graph on k vertices, each vertex with a loop of weight y attached to it. By the identity $Z(G) = e^{-K|E|}B(G; q, e^K)$ it follows that the partition function $Z(G)$ for the q-state Potts model is the number of homomorphisms from G to the complete graph on q vertices , each vertex with a loop of weight 1 and non-loop edges of weight e^{-K}.

Another example of a homomorphism counting function of interest to statistical physics is the Widom-Rowlinson model (introduced in 1969 as a model for liquid-vapour phase transitions), where the target graph consists of a star $K_{1,k}$ with a loop of weight 1 on each vertex. The number of homomorphisms from G to this graph is equal to the number of partial k-colourings of the vertices of G with the property that no edge has an endpoint of different colours (but it is allowed to have one endpoint a coloured vertex and the other uncoloured).

Amongst all possible weighted graphs H, the number of homomorphism from G to H is an evaluation of the Tutte polynomial for every graph G if and only if H is a Potts model graph [15], [14]. (A Potts model graph is K_q with a constant weight on its edges, together with loops attached, also of constant weight.) In fact, given just that the number of graph homomorphisms from G to H is an evaluation of the Tutte polynomial for G a cycle or path or the dual of a cycle or path, it must be the case that H is a Potts model graph.

As we have seen, the partition function of the Potts model is the specialization of the Tutte polynomial to the hyperbola $(x-1)(y-1)=q$. In [14] it is shown that any evaluation of the Tutte polynomial can be interpolated from its values on the hyperbolae $(x-1)(y-1)=q$ for positive integer q. For a familiar example, the number of acyclic orientations, $T(G;2,0)$, the point $(2,0)$ lying on $(x-1)(y-1)=-1$, can be found by interpolation from the values $T(G;1-q,0)$ for $r(G)+1$ choices of positive integer q, the points $(1-q,0)$ lying on the hyperbolae $(x-1)(y-1)=q$. In this sense, the partition functions of the q-state Potts model for all positive integers q contain all the information about a graph that the Tutte polynomial does. What about when only finitely many values of q are chosen? Although it seems likely that a finite number of Potts model partition functions will not determine the Tutte polynomial in general, it seems difficult to produce examples of a pair of graphs that have different Tutte polynomials but the same q-state Potts model for even a fixed value of $q \geq 3$. (For $q=2$ there are small examples of graphs with the same Ising model partition function but different Tutte polynomials.)

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