Mathematical Skills

Exercise sheet

7 December 2015

A statement $P(n)$ about a natural number n can be proved to hold for all $n \geq n_0$ (usually $n_0 = 1$ or $n_0 = 0$) by

(i) (*Induction basis*) Verifying $P(n_0)$ is true, and

(ii) (*Induction step*) Proving that $P(n) \Rightarrow P(n+1)$.

In the induction step (ii) the *induction hypothesis* is that $P(n)$ is true. While the hypothesis is not known to be true for general n yet, the basis for induction gives that $P(n_0)$ is true. Hence $P(n_0+1)$ is true by (ii), and so $P(n_0+2)$ is true, again by (ii), etc. In other words the truth of $P(n_0)$ is transmitted by the implication in (ii) to the truth of $P(n)$ for all natural numbers $n \geq n_0$.

1. Prove the following statements by mathematical induction:

(i) $n! > 2^n$ for $n \geq 4$.

(ii) $(1+x)^n \geq 1 + nx$ for $n \geq 0$ and real number $x > -1$.

(iii) $\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \binom{n+1}{2}$ for $n \geq 1$.

(iv) For $n \geq 1$, the number of subsets of $\{1, 2, \ldots, n\}$ (including the empty set) is 2^n.

(v) For all $n \geq i \geq 0$,

$$\sum_{j=i}^{n} \binom{j}{i} = \binom{n+1}{i+1},$$

(where we set $\binom{0}{0} = 1$).

2. Either prove by induction or by some other method the following statements:

(i) For $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

(ii) For all $n \geq 1$,

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}.$$

(iii) For all $n \geq 0$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

(iv) For all $n \geq 1$,

$$\sum_{i=1}^{n} i = \binom{n+1}{2}.$$

(v) For $n \geq 0$,

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$