Mathematical Analysis I

Exercise sheet 8

Solutions to selected exercises

3 December 2015

References: Abbott 4.2, 4.3. Bartle & Sherbert 4.1, 4.2, 5.1, 5.2

4. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sqrt[3]{x}$.

(ii) Show that g is continuous at $c = 0$. We have $|\sqrt[3]{x} - \sqrt[3]{0}| = \sqrt[3]{|x|} < \epsilon$ when $|x - 0| = |x| < \epsilon^3$.

(iii) Prove that g is continuous at a point $c \neq 0$. Take first $c > 0$. Then for $x > 0$, using the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, we have

$$|\sqrt[3]{x} - \sqrt[3]{c}| = \frac{|x - c|}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}}$$

in which the denominator on the right-hand side is bounded below by $\sqrt[3]{c^2}$. Hence,

$$|\sqrt[3]{x} - \sqrt[3]{c}| < \frac{|x - c|}{\sqrt[3]{c^2}}$$

and taking $\delta = \min\{\sqrt[3]{c^2}\epsilon, c\}$ we have $|\sqrt[3]{x} - \sqrt[3]{c}| < \epsilon$ when $|x - c| < \delta$. (We required $x > 0$ to apply the bound on the denominator above, hence this condition that $x - c > -c$ is incorporated into $|x - c| < \delta$ by making sure $\delta \leq c$.)

When $c < 0$ use the fact that $\sqrt[3]{c} = -\sqrt[3]{-c}$ and use continuity of the cube root at $-c > 0$ to deduce continuity at c.

(iv) Assuming the result of question 3(iv), deduce that $\sqrt[3]{p(x)}$ is continuous on $\mathbb{R}$ for any polynomial $p(x)$ with real coefficients.

Question 3(iv) states that a polynomial $p(x)$ with real coefficients is continuous at c for any $c \in \mathbb{R}$. By applying the first part of this question (the composition of continuous functions is continuous) to $p : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sqrt[3]{x}$, we deduce that the composition $g \circ p$ is continuous on $\mathbb{R}$.

5. For each of the following choices of A, construct a a function $f : \mathbb{R} \to \mathbb{R}$ which has discontinuities at every point of A and is continuous on the complement $\mathbb{R} \setminus A$:

(i) $A = \mathbb{Z}$

Define $f : \mathbb{R} \to \mathbb{Z}$ by $f(x) = \lfloor x \rfloor$, the greatest integer less than or equal to x. Thus $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

For $z \in \mathbb{Z}$, the sequence (x_n) defined by $x_n = z - \frac{1}{n}$ converges to z while $(f(x_n))$ converges to $z - 1 \neq f(z) = z$, since $f(x_n) = z - 1$ for all n.

On the other hand, for $c \in \mathbb{R}\setminus\mathbb{Z}$ there is $z \in \mathbb{Z}$ such that $z < c < z+1$. Set $\delta = \min\{c-z, z+1-c\}$. Then $f(x) = f(c)$ for $|x - c| < \delta$, and so $|f(x) - f(c)| < \epsilon$ for any given $\epsilon > 0$ when $|x - c| < \delta$. This says f is continuous at c.

(ii) $A = \{x : 0 < x < 1\}$

For (ii) and (iii) we shall use as a building block the *Dirichlet function* $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not continuous at any point in $\mathbb{R}$. (Proof sketch: use density of $\mathbb{Q}$ and of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$ to show that for any $c \in \mathbb{R}$ there are sequences (a_n) of rationals convergent to c and sequences (b_n) of irrationals (b_n) also convergent to c.) Also useful is the modified Dirichlet function

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

which is continous at 0 and nowhere else. See Abbott §4.1 for a discussion of these functions and Thomae's function (continous precisely at irrational points).

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & x \in \mathbb{Q}, 0 < x \leq \frac{1}{2} \\ 1 - x & x \in \mathbb{Q}, \frac{1}{2} < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

is not continous on $\{x : 0 < x < 1\}$ (for the same reason as the modified Dirichlet function on $\mathbb{R}$) but is continuous outside this interval ($f(x) \to 0 = f(c)$ as $x \to c$ when $c \leq 0$ or $c \geq 1$).

(iii) $A = \{x : 0 \leq x < 1\}$ The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \leq 0, \\ x & x \in \mathbb{Q}, 0 < x < 1 \\ 1 & \text{otherwise,} \end{cases}$$

is not continuous on $\{x : 0 \leq x < 1\}$ (due to density of irrationals in this interval, where f takes the value 1) but is continuous outside this interval ($f(x) \to 0 = f(c)$ as $x \to c$ when $c < 0$ and $f(x) \to 1 = f(c)$ when $c \geq 1$).

(iv) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ One example is the function

$$f(x) = \begin{cases} \lfloor \frac{1}{x} \rfloor & x \geq 1 \\ 0 & x < 1, \end{cases}$$

is discontinuous at points $\frac{1}{n}$ (see part (i)) and continuous elsewhere.

6. Let $f : \mathbb{R} \to \mathbb{R}$ and assume there is a constant C such that $0 < C < 1$ and

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in \mathbb{R}$. Let $f^n(x)$ be inductively defined by $f^1(x) = f(x)$, and $f^{n+1}(x) = f(f^n(x))$. (We could start from $f^0(x) = x$.) It is useful to first prove by induction the inequality

$$|f^n(x) - f^n(y)| \leq C^n|x - y|.$$

For $n = 1$ it is the inequality given in the question, and the inductive step is

$$|f^{n+1}(x) - f^{n+1}(y)| \leq C|f^n(x) - f^n(y)| \leq C \cdot C^n|x - y| = C^{n+1}|x - y|.$$

(i) Show that f is continuous on $\mathbb{R}$. When $|x-c| < \epsilon/C$ we have

$$|f(x) - f(c)| \le C|x - c| < \epsilon.$$

Hence f is continuous at any point $c \in \mathbb{R}$.

(ii) Beginning with an initial value $y_1 \in \mathbb{R}$, define the sequence $(y_n) = (y_1, f(y_1), f(f(y_1)), \dots)$ recursively by setting $y_{n+1} = f(y_n)$. Show that (y_n) is a Cauchy sequence.

In the notation introduced above, $y_n = f^{n-1}(y_1)$.

For $m \ge n \ge 1$,

$$\begin{aligned}
|y_m - y_n| &= |f^{m-1}(y_1) - f^{n-1}(y_1)| \\
&\le C^{n-1}|f^{m-n}(y_1) - y_1| \\
&\le C^{n-1}(|f^{m-n}(y_1) - f^{m-n-1}(y_1)| + |f^{m-n-1} - f^{m-n-2}| + \cdots + |f(y_1) - y_1|) \\
&\le C^{n-1}(C^{m-n-1} + C^{m-n-2} + \cdots C + 1)|f(y_1) - y_1| \\
&< C^{n-1}\sum_{k=0}^{\infty} C^k |y_2 - y_1| \\
&= \frac{C^{n-1}}{1-C}|y_2 - y_1|
\end{aligned}$$

Since $(C^{n-1}) \to 0$ (because $0 < C < 1$) and $\frac{|y_2 - y_1|}{1-C}$ is constant, we deduce that $(y_n) = (f^{n-1}(y_1))$ is a Cauchy sequence (for any given $\epsilon > 0$ we can choose N such that $|y_m - y_n| < \epsilon$ for $m, n \ge N$).

(iii) Let $y = \lim y_n$. Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is the unique fixed point of f (i.e., if $f(y') = y'$ then $y' = y$).

By (ii) the sequence (y_n) is convergent to some limit y. Continuity of f implies that

$$f(y) = f(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} f(y_n) = \lim y_{n+1} = y.$$

Suppose y and y' are fixed points, i.e., $y = f(y)$ and $y' = f(y')$. Then

$$0 \le |y - y'| = |f(y) - f(y')| \le C|y - y'|$$

and since $0 < C < 1$ this forces $|y - y'| = 0$, i.e., $y = y'$.

(iv) For an arbitrary initial value $x \in \mathbb{R}$, show that the sequence $(x_n) = (x, f(x), f(f(x)), \dots)$ defined recursively by $x_{n+1} = f(x_n)$ is convergent to the value y defined in (iii).

In the notation introduced above, $x_{n+1} = f^n(x)$ and

$$\begin{aligned}
|f^n(x) - y| &= |f(f^{n-1}(x) - f(y)| \\
&\le C|f^{n-1}(x) - y|
\end{aligned}$$

and by induction on n

$$|f^n(x) - y| \le C^n|x - y|.$$

(Base $n = 1$ is $|f(x) - y| = |f(x) - f(y)| \le C|x - y|$. Inductive step is $|f^{n+1} - y| = |f(f^n(x)) - f(y)| \le C|f^n(x) - y| \le C \cdot C^n|x - y| = C^{n+1}|x - y|$.)

Hence $f^n(x) \to y$ as $n \to \infty$, since $C^n \to 0$ as $n \to \infty$, i.e., $(x_n) \to y$.

[The result of this question is known as the Contraction Mapping Theorem, or Banach's Fixed Point Theorem.]