Mathematical Analysis I

Exercise sheet 7

26 November 2015

References: Abbott 2.1, 2.8, 2.9, 3.2.

1. Let the function $\exp : \mathbb{R} \to \mathbb{R}$ be defined by the series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{1}$$

in which $0! = 1 = x^0$ and $(n+1)! = (n+1) \cdot n!$ for $n \geq 0$. In particular, $\exp(0) = 1$.

(i) Show that the series in equation (1) converges for all $x \in \mathbb{R}$.

(ii) Show that the function exp satisfies the identity

$$\exp(x+y) = \exp(x)\exp(y),$$

for all $x, y \in \mathbb{R}$.

(iii) Let $\exp(1) = e$. Prove that $\exp(a) = e^a$ for $a \in \mathbb{N}$.

(iv) Show that $\exp(-a) = e^{-a}$ for $a \in \mathbb{N}$.

(v) Prove that $\exp(\frac{a}{b}) = e^{\frac{a}{b}}$ for $\frac{a}{b} \in \mathbb{Q}$.

[*Later it will be shown that if* $(\frac{a_n}{b_n})$ *is a sequence of rationals convergent to* $x \in \mathbb{R}$ *then* $\lim_{n\to\infty} \exp(\frac{a_n}{b_n}) = e^x$ *and that* $\exp(x) = e^x$ *for all* $x \in \mathbb{R}$.]

2. The ϵ-neighbourhood of $a \in \mathbb{R}$ is the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

(i) Let $A \subseteq \mathbb{R}$. Define what is meant for A to be an *open set*, for $x \in \mathbb{R}$ to be a *limit point* of A, and what it means for A to be a *closed* set.

(ii) Prove that x is a limit point of A if and only if there is a sequence (a_n) of points $a_n \in A \setminus \{x\}$ converging to x.

(iii) Prove that $\mathbb{Q}$ contains every $x \in \mathbb{R}$ as a limit point.

3. Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(i) $\mathbb{Q}$

(ii) $\mathbb{N}$

(iii) $\{x \in \mathbb{R} : x > 0\}$

(iv) $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$

(v) $\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\}$.

4. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true. ($\overline{A}$ denotes the closure of A, i.e., the union of A and its limit points.)

(i) For any set $A \subseteq \mathbb{R}$, the set $\mathbb{R} \setminus \overline{A}$ is open.

(ii) If a set A has an isolated point, it cannot be an open set.

(iii) A set A is closed if and only if $\overline{A} = A$.

(iv) If A is a bounded set, then $s = \sup A$ is a limit point of A.

(v) Every finite set is closed.

(vi) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

5. The *Fibonacci sequence* (f_n) is defined recursively by $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$, with initial values $f_0 = 0, f_1 = 1$. Define the function $F(x)$ by the power series

$$F(x) = \sum_{n=0}^{\infty} f_n x^n.$$

(i) By summing the identity

$$f_{n+1}x^{n+1} = f_n x^{n+1} + f_{n-1}x^{n+1}$$

over $n \geq 1$, prove that

$$F(x) = \frac{x}{1 - x - x^2}.$$

(ii) Show that $F(x)$ has the partial fraction expansion

$$F(x) = \frac{\frac{1}{\sqrt{5}}}{1 - \tau x} - \frac{\frac{1}{\sqrt{5}}}{1 - (1 - \tau)x}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, Write down the radius of convergence of the power series that was used to define $F(x)$.

(iii) Deduce from (ii) that

$$f_n = \frac{1}{\sqrt{5}}(\tau^n - (1 - \tau)^n).$$

(iv) Show that f_n is the nearest integer to $\frac{1}{\sqrt{5}}\tau^n$.