Mathematical Analysis I

Exercise sheet 6

12 November 2015

References: Abbott 2.7. Bartle & Sherbert 3.7

1. Define what it means for an infinite series $\sum_{n=1}^{\infty} a_n$ to converge.

   (i) Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $(a_n) \to 0$.

   (ii) Give a counterexample to the converse of (i).

   (iii) Let $r \in \mathbb{R}$. Prove that the series $\sum_{n=1}^{\infty} r^n$ converges if and only if $(r^n) \to 0$, and write down its limit in this case.

2.

   (i) Let (a_n) be a sequence of nonnegative reals. Prove that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence of its partial sums $(a_1 + \cdots + a_n)$ is bounded.

   (ii) Using the result of (i), prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

   (iii) Deduce from (ii) that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \leq 1$.

   (iv) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$. [*If you use the Cauchy Condensation Test here you must prove it, so you may wish to argue directly instead.*]

3. Let (a_n) be a sequence of reals that is monotone decreasing and converges to 0.

   (i) Prove that the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to a limit between $a_1 - a_2$ and a_1. [*Two methods of proving this are (1) to use the Cauchy Criterion for series, or (2) to apply the Monotone Convergence Theorem to the two sequences of partial sums* $(a_1 - a_2 + \cdots - a_{2n})$ *and* $(a_1 - a_2 + \cdots - a_{2n} + a_{2n+1})$.]

   (ii) Deduce from (i) that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges for $p > 0$. [*In particular the alternating harmonic series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges to a limit between* $\frac{1}{2}$ *and* 1 *(the limit, as you will later prove, is* $\ln 2$*).*]

4. For each of the following series, prove either that it diverges, or that it converges to a limit and in this case determine the limit:

   (i) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

   (ii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

   (iii) $\sum_{n=1}^{\infty} \frac{1}{n \sqrt[n]{n}}$

   (iv) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)}$

   (v) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

5. Let (a_n) be a sequence of strictly positive reals and suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. Either prove or give a counterxample to the following statements:

(i) the series $\sum_{n=1}^{\infty} a_n^2$ converges,

(ii) the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges,

(iv) the series $\sum_{n=1}^{\infty} \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$ diverges.

6. Define what is meant by a *rearrangement* of a series $\sum_{n=1}^{\infty} a_n$.

(i) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges then so does $\sum_{n=1}^{\infty} a_n$.

(ii) Prove further that if $\sum_{n=1}^{\infty} |a_n|$ converges then any rearrangement of the series $\sum_{n=1}^{\infty} a_n$ converges to the same limit.

(iii) Prove that if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges then, for any $l \in \mathbb{R}$, there is some rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to l.

[*This result is part of what is known as* Riemann's Series Theorem. *For simplicity you may assume $a_n \neq 0$ for each $n \in \mathbb{N}$ (this does not affect the result in the end, but helps define the construction needed in the proof more smoothly). Define $a_n^+ = \max\{0, a_n\}$ and $a_n^- = \min\{0, a_n\}$. Since $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n^+$ diverges to $+\infty$ and the series $\sum_{n=1}^{\infty} a_n^-$ diverges to $-\infty$. Use partial sums of these two series to construct a rearrangement of $\sum_{n=1}^{\infty} a_n$ that is convergent to l.*]