Mathematical Analysis I

Exercise sheet 5

Solutions to selected exercises

5 November 2015

References: Abbott 2.5, 2.6. Bartle & Sherbert 3.3, 3.4, 3.5

2. Show that if a sequence of real numbers (a_n) has either of the following properties then it is divergent:

(i) (a_n) has two subsequences that converge to different limits,

(ii) (a_n) is unbounded.

Is the converse true, that any divergent sequence is either unbounded or has two subsequences that converge to a different limit?

We use the fact that a sequence (a_n) is convergent to a limit l if and only if every subsequence of (a_n) is convergent to the same limit l.

Taking the negation of each side of this equivalence, we have:[1]

A sequence is divergent if and only if for every $l \in \mathbb{R}$ there is some subsequence of (a_n) not convergent to l.

A subsequence (a_{n_i}) of (a_n) can fail to converge to l in two different ways: either (a_{n_i}) has a subsequence that converges to a limit $l' \neq l$, or (a_{n_i}) is unbounded. The latter puts us in case (ii). In the former case, when a subsequence of (a_n) is unbounded the whole sequence (a_n) is also unbounded. Else we obtain two subsequences with distinct limits, putting us in case (i).

Proof of former claim,[2] that if all subsequences of a divergent sequence (a_n) that converge have the same limit l' then there must be some unbounded subsequence of (a_n).

If all subsequences of (a_n) were bounded then (a_n) would itself be bounded, in which case a divergent subsequence of (a_n) must contain two subsequences convergent to different limits. To see this, suppose to the contrary that (a_n) is a divergent bounded sequence that does not have two convergent subsequences with different limits. By the Bolzano-Weierstrass Theorem (a_n) has a subsequence convergent to a limit l. Now use the fact that l is not the limit of some subsequence of (a_n) (since (a_n) is divergent by hypothesis) to produce a subsequence of (a_n) bounded away from l – i.e., belonging to some bounded interval disjoint from a neighbourhood of l. Applying the Bolzano-Weierstrass Theorem again, we produce a convergent subsequence of (a_n) that has a limit that must be different to l. This contradicts the assumption that any convergent subsequence has the same limit.

4.

(i) Let $c > 0$ be a positive real number. Use the Monotone Convergence Theorem to show that the sequence $(c^{\frac{1}{n}})$ is convergent and determine its limit. [*Consider the cases* $0 < c < 1$ *and* $c > 1$ *separately.*]

[1] In terms of logic we are using the fact that $P \Leftrightarrow Q$ is logically equivalent to $\neg P \Leftrightarrow \neg Q$. Here Q is the statement "there exists l such that for every subsequence of (a_n) we have $(a_n) \to l$." The negation is "for every l there is some subsequence of (a_n) which does not converge to l."

[2] An alternative proof is to use question 5(ii) to deduce directly that the whole sequence (a_n) is convergent to limit l', contrary to hypothesis.

Write $a_n = c^{\frac{1}{n}}$. We will show that (a_n) is monotone and bounded, and so will be able to apply the Monotone Convergence Theorem (MCT).

Consider first $0 < c < 1$. Then $0 < c^{\frac{1}{n}} < 1$ for all n (since exponentation to a positive number is an increasing function, preserving inqualities). So (a_n) is bounded below by 0 and above by 1.

Then $c^{\frac{n+1}{n}} = c \cdot c^{\frac{1}{n}} < c$, whence $a_{n+1} = c^{\frac{1}{n+1}} < c^{\frac{1}{n}} = a_n$, so (a_n) is strictly increasing.

MCT implies that (a_n) converges to some limit l with $0 < l \le 1$. The subsequence $(a_{2n}) = (c^{\frac{1}{2n}}) = (a_n^{\frac{1}{2}})$ must converge to the same limit l. By question 3(i) we know that $(a_n^{\frac{1}{2}}) \to l^{\frac{1}{2}}$. This gives $l = l^{\frac{1}{2}}$, which together with $l > 0$ implies $l = 1$. Thus $(c^{\frac{1}{n}}) \to 1$.

Consider now the sequence $(a_n) = (c^{\frac{1}{n}})$ for $c > 1$. Here $a_n = c^{\frac{1}{n}} > 1$, so (a_n) is bounded below.

With $c^{\frac{n+1}{n}} = c \cdot c^{\frac{1}{n}} > c$, we have $a_n = c^{\frac{1}{n}} > c^{\frac{1}{n+1}} = a_{n+1}$, so (a_n) is strictly decreasing.

By MCT (a_n) converges to a limit $l \ge 1$. Since the subsequence $(a_{2n}) = (a_n^{\frac{1}{2}})$ converges to the same limit we must have again $l^{frac12} = l$, whence $l = 1$.

(ii) Is the sequence $(n^{\frac{1}{n}})$ convergent? Either give a proof of divergence or, if it is convergent, determine the limit of the sequence.

Now set $a_n = n^{\frac{1}{n}}$. We have $a_n > 1$ for all $n \ge 1$.

We show that (a_n) is decreasing for $n \ge 3$ by showing that $\frac{a_{n+1}}{a_n} < 1$ for $n \ge 3$. It is easier to claim the equivalent inequality $\left(\frac{a_{n+1}}{a_n}\right)^{n(n+1)} < 1$, which is to say $\frac{(n+1)^n}{n^{n+1}} < 1$, for $n \ge 3$. To prove this claim, rewrite the inequality as

$$\left(1 + \frac{1}{n}\right)^n < n$$

and use the following

Lemma We have $\left(1 + \frac{1}{n}\right)^n < 3$ for all $n \in \mathbb{N}$.

Proof By the binomial expansion,

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) \\
&\le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&\le 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \\
&= 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 3 - \frac{1}{2^{n-1}}
\end{aligned}$$

where in the penultimate line we used the fact that $2^{n-1} \le n!$ so that $\frac{1}{n!} \le \frac{1}{2^{n-1}}$, and in the last line we summed a geometric series. $\square$

Using the above Lemma, we have $(1 + \frac{1}{n})^n < 3$, whence $(1 + \frac{1}{n})^n < n$ for $n \ge 3$, proving the claim. This yields the desired conclusion that (a_n) is decreasing for $n \ge 3$. Since (a_n) is bounded below by 1 we conclude by MCT that (a_n) converges to a limit $l \ge 1$. By considering the subsequence $(a_{2n}) = ((2n)^{\frac{1}{2n}})$, also convergent to l, but also to $\lim(\sqrt{2}^{\frac{1}{n}} n^{\frac{1}{2n}}) = \lim \sqrt{2}^{\frac{1}{n}} \lim n^{\frac{1}{2n}}$ by the algebra of limits applied to the convergent sequences $(\sqrt{2}^{\frac{1}{n}})$ and $(n^{\frac{1}{2n}})$, we have $l = 1 \cdot l^{\frac{1}{2}}$, using part (i) of this question with $c = \sqrt{2}$ and question 3(i). Thus $l = l^{\frac{1}{2}}$ and along with $l \ge 1$ this forces $l = 1$.

Hence $\lim n^{\frac{1}{n}} = 1$.

5.

(i) Prove that a sequence of reals (a_n) has a monotone subsequence. Deduce that a bounded sequence of reals has a convergent subsequence (Bolzano–Weierstrass Theorem).

Define a term a_m of (a_n) to be a *peak* if $\forall n \geq m \quad a_m \geq a_n$ (a_m is higher than any later term).

There are two case to consider.

(1) There are infinitely many peaks. Here we have a subsequence (a_{m_i}) consisting of peaks, which must be monotone decreasing since by definition of a peak $a_{m_i} \geq a_n$ for all $n \geq m_i$, in particular for $n = m_{i+1}$.

(2) There are finitely many peaks. Let the peaks be $a_{m_1}, \ldots, a_{m_k}$. Set $n_1 = m_k + 1$. Since a_{n_1} is not a peak there is some $n_2 > n_1$ such that $a_{n_1} < a_{n_2}$. Since a_{n_2} is not a peak, there is $n_3 > n_2$ such that $a_{n_2} < a_{n_3}$. Continuing in this way we obtain a subsequence (a_{n_i}) that is monotone increasing.

Now suppose that (a_n) is a bounded sequence.. Then by the previous (a_n) has a monotone subsequence, which must also be bounded. By MCT this subsequence converges to a limit.

(ii) Prove as a corollary of (i) that if a bounded sequence of reals (a_n) has the property that every subsequence that is convergent has the same limit l, then the whole sequence is itself convergent to l.

Let (a_n) be a bounded sequence with the property that any subsequence that converges has the same limit l. (Note that it is not asssumed that every subsequence converges.)

Suppose to the contrary that (a_n) does not converge to l. Then $\exists \epsilon > 0 \forall N \exists n \geq N \quad |a_n - l| \geq \epsilon$. We can thus for some $\epsilon > 0$ form a subsequence (a_{n_i}) such that $|a_{n_i} - l| \geq \epsilon$ for all i. The subsequence (a_{n_i}) is bounded since (a_n) is. By (i) (Bolzano–Weierstrass Theorem) (a_{n_i}) has a convergent subsequence, which is also then a subsequence of (a_n). By hypothesis this subsequence of (a_{n_i}) converges to l, contradicting the fact that $|a_{n_i} - l| \geq \epsilon$ for all i. Hence our assumption that (a_n) does not converge to l was false, and so $(a_n) \to l$.

(iii) Give an example to show that the condition in (ii) that (a_n) is bounded is necessary.

Any example of an unbounded sequence interleaved with a convergent sequence, e.g. $1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots$.

6. Define what it means for a sequence of reals (a_n) to be a *Cauchy sequence.*

The sequence (a_n) is a Cauchy sequence if $\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall m, n \geq N \quad |a_m - a_n| < \epsilon$.

(i) Suppose (a_n) is a sequence with the property that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$. Is (a_n) necessarily a Cauchy sequence? (Give a counterexample if not, a proof if so.)

Take $a_n = \sqrt{n}$. Then $a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$ as $n \to \infty$ but for example $a_{2n} - a_n = \sqrt{2n} - \sqrt{n} = (\sqrt{2} - 1)\sqrt{n} \to \infty$ as $n \to \infty$, so (a_n) is not a Cauchy sequence.

(ii) A sequence (a_n) is *contractive* if there is a constant C with $0 < C < 1$ such that

$$|a_{n+2} - a_{n+1}| \leq C|a_{n+1} - a_n|$$

for all $n \in \mathbb{N}$. Prove that a contractive sequence is a Cauchy sequence.

Repeatedly applying the contractive inequality, we have

$$|a_{n+2} - a_{n+1}| \leq C|a_{n+1} - a_n| \leq C^2|a_n - a_{n-1}| \leq \cdots \leq C^n|a_2 - a_1|.$$

For $m > n$, by the triangle inequality we have

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&\leq (C^{m-2} + C^{m-3} + \cdots + C^{n-1})|a_2 - a_1| \\
&= C^{n-1}\frac{1 - C^{m-n}}{1 - C}|a_2 - a_1| \\
&< C^{n-1}\frac{1}{1 - C}|a_2 - a_1| \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

since $0 < C < 1$ so that $(C^n) \to 0$. Hence by taking $m > n$ sufficiently large the difference $|a_m - a_n|$ can be made arbitrarily small, i.e., (a_n) is a Cauchy sequence.

(iii) Show that the sequence (a_n) defined recursively by $a_{n+1} = (2 + a_n)^{-1}$ is contractive when $a_1 > 0$ and determine its limit.

Set $a_{n+1} = \frac{1}{2+a_n}$. If $a_1 > 0$ then $a_n > 0$ for each n (by induction). We have

$$|a_{n+2} - a_{n+1}| = \left| \frac{1}{2 + a_{n+1}} - \frac{1}{2 + a_n} \right| = \frac{|a_{n+1} - a_n|}{(2 + a_n)(2 + a_{n+1})},$$

and since $2 + a_n > 2$ and $2 + a_{n+1} > 2$ we have $|a_{n+2} - a_{n+1}| < \frac{1}{4}|a_{n+1} - a_n|$, so that (a_n) is contractive with constant $C = \frac{1}{4}$.

Thus we know (a_n) is convergent to a limit l, since it is a Caucht sequence. By the algebra of limits applied to the equation $a_{n+1} = \frac{1}{2+a_n}$ we obtain $l = \frac{1}{2+l}$, when $l^2 + 2l - 1 = 0$. This quadratic has roots $-1 \pm \sqrt{2}$. Since $a_n > 0$ we have $l \geq 0$, and thus $l = \sqrt{2} - 1$.