Mathematical Analysis I

Exercise sheet 3

Solutions to selected exercises

22 October 2015

References: Abbott, 1.3, 1.4 and 8.4. Bartle & Sherbert 1.3, 2.3, 2.4, 2.5

1. Define a *Dedekind cut* of the rationals. Fix $r \in \mathbb{Q}$. Show that the set $C_r = \{x \in \mathbb{Q} : x < r\}$ is a Dedekind cut.

A subset $A \subseteq \mathbb{Q}$ is a Dedekind cut if

(1) $A \neq \emptyset$ and $A \neq \mathbb{Q}$

(2) $\forall a \in A \ \forall x \in \mathbb{Q} \quad (x < a \ \Rightarrow \ x \in A)$

(3) $\forall a \in A \ \exists x \in A \quad (a < x)$ (A has no maximum)

A cut is bounded above:

(4) $\exists x \in \mathbb{Q} \ \forall a \in A \quad (a < x)$.

[Proof: suppose not, then $\forall x \in \mathbb{Q} \ \exists a \in A \quad x \leq a$, which by property (2) implies $\forall x \in \mathbb{Q} \ x \in A$, which is to say that $A = \mathbb{Q}$, contrary to property (1).]

The set $C_r = \{x \in \mathbb{Q} : x < r\}$ is non-empty and not all of $\mathbb{Q}$, and for any $a \in C_r$ and $x \in \mathbb{Q}$ with $x < a$ we have $x \in C_r$ since $x < a < r$ implies $x < r$. Finally, C_r has no maximum since for any given $a < r$ we may take $x = \frac{1}{2}(a + r) < r$, which also belongs to C_r.

3. For $A, B \subseteq \mathbb{R}$ define
$$A + B = \{a + b : a \in A, b \in B\}.$$

(i) Show that if A and B are Dedekind cuts then so is $A + B$.

Property (1): If $A, B \notin \{\emptyset, \mathbb{Q}\}$ then clearly $A + B \neq \emptyset$. Also, $A + B \neq \mathbb{Q}$ since A and B are bounded above by (4), so $A + B$ is also bounded above.

Property (2): take an arbitrary element $a + b \in A + B$ and for any $x \in \mathbb{Q}$ with $x < a + b$ we have $x - b < a$ so that $x - b \in A$ (by property (2) for cut A) amd then $x = (x - b) + b \in A + B$, which shows that (2) holds for $A + B$.

Property (3): by property (3) for cuts A and B, for any $a \in A$ and $b \in B$ there are $x \in A$ and $y \in B$ such that $a < x$ and $b < y$. By addition of these inequalities, we have $a + b < x + y$, and also $x + y \in A + B$.

(ii) Let $A = \{x \in \mathbb{Q} : x < a\}$ and $B = \{x \in \mathbb{Q} : x < b\}$ for $a, b \in \mathbb{Q}$. From question 1 we know A and B are Dedekind cuts. What cut is $A + B$?

Let $A = C_a$ and $B = C_b$. We show that $C_a + C_b = C_{a+b}$.

$C_a + C_b \subseteq C_{a+b}$: If $x \in C_a, y \in C_b$ then $x < a$ and $y < b$, adding together which give $x + y < a + b$, so $x + y \in C_{a+b}$.

$C_a + C_b \supseteq C_{a+b}$: If $z \in C_{a+b}$ then $z - b < a$ so by property (2) for C_a there is $x \in C_a$ such that $z - b < x < a$, whence $z < a + b$, i.e., $z \in C_a + C_b$.

(iii) Define the Dedekind cut $O = \{r \in \mathbb{Q} : r < 0\}$. Show that O is an identity for addition of cuts and write down the inverse to a cut A with respect to this operation.

Let $O = \{r \in \mathbb{Q} : r < 0\}$. We prove that $A + O = A$ for any cut A.

$A \supseteq A + O$: taking arbitrary $a \in A$ and $z \in O$, we have $a + z < a + 0 = a$, whence $a + z \in A$.

$A \subseteq A+O$: if $a \in A$ then there is by property (2) for cuts $x \in A$ such that $a < x$. Then $a-x < 0$, so that by definition of O there is $z \in O$ such that $z = a - x$, whence $a = x + z \in A + O$.

The additive inverse to A is the cut defined by

$$-A = \{x \in \mathbb{Q} : \exists y \notin A \;\; y < -x\}.$$

(See the diagram in Abbott, p. 247.)*

Lemma: The cut $-A$ has the property that if $a \in A$ then $-a \in -A$.

Proof. Suppose not. Then for all $y \notin A$ we have $y \geq a$. Hence $A \subseteq \{x \in \mathbb{Q} : x \leq a\}$. But $a \in A$ is then a maximum for A, contradicting property (3) for the cut A. $\square$

We show that $A + (-A) = O$.†

$A+(-A) \subseteq O$: If $a \in A$ and $x \in -A$ then there is $y \notin A$ such that $-y > x$, whence $a+x < a-y$ and $a < y$ since $y \notin A$ (by property (2), if $y \leq a$ and $a \in A$ then $y \in A$). Hence $a + x < 0$ and so $a + x \in O$.

$A + (-A) \supseteq O$: if $z \in O$ then $z < 0 = a + (-a)$ for any $a \in A$. By the Lemma above we have $-a \in -A$. This implies $z \in A + (-A)$ by property (2) for the cut O.

*Faulty choices for defining the inverse include $-A = \{-x : x \in A\}$ (not a cut as it does not satisfy property (2) – this definition of $-A$ "reflects" A about the 0 point of the rational line) and $-A = \{x \in \mathbb{Q} : -x \notin A\}$ (not a cut as it contains a maximum, violating property (3) – the correct definition remedies this by excluding the possibility of a maximum.

†This was not asked in the question, but is here for purposes of edification.