Mathematical Analysis I

Exercise sheet 1

Solutions to selected exercises

8 October 2015

3. For a function $f : X \to Y$ and $A \subseteq X$ we define $f(A) = \{f(x) : x \in A\}$. Thus $f(X)$ is the range of f with domain X.

(ii) Show that if $f : X \to Y$ and $A, B \subseteq X$ then $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) \subseteq f(A) \cap f(B)$.

By definition $A \cup B = \{x : x \in A \vee x \in B\}$. We have then

$$y \in f(A \cup B) \quad \Leftrightarrow \quad y \in \{f(x) : x \in A \vee x \in B\} \tag{1}$$

and

$$y \in f(A) \cup f(B) \quad \Leftrightarrow \quad y \in \{f(x) : x \in A\} \ \vee \ y \in \{f(x) : x \in B\}. \tag{2}$$

We have

$$\begin{aligned}
y \in \{f(x) : x \in A \vee x \in B\} &\Leftrightarrow \quad \exists x\, (y = f(x)) \wedge [(x \in A) \vee (x \in B)] \\
&\Leftrightarrow \quad \exists x\, [(y = f(x)) \wedge (x \in A)] \quad \vee \quad [(y = f(x)) \wedge (x \in B)] \\
&\Leftrightarrow \quad \exists x\, [(x \in A) \wedge (y = f(x))] \quad \vee \quad \exists x\, [(x \in B) \wedge (y = f(x))] \\
&\Leftrightarrow \quad \exists x \in A\, [y = f(x)] \quad \vee \quad \exists x \in B\, [y = f(x)] \\
&\Leftrightarrow \quad y \in \{f(x) : x \in A\} \ \vee \ y \in \{f(x) : x \in B\}
\end{aligned}$$

In moving from the first line to the second line we used distributivity of $\wedge$ over $\vee$, i.e., $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$.

For the second to the third line we used $\exists x\, [P(x) \vee Q(x)] \ \Leftrightarrow \ \exists x P(x) \ \vee \ \exists x\, Q(x)$. (However, it is not the case that $\exists x\, [P(x) \wedge Q(x)] \ \Leftrightarrow \ \exists x P(x) \ \wedge \ \exists x\, Q(x)$ - take for example $P(x) = \neg Q(x)$ and the former statement is false, while the latter can be true, for example if $P(x)$ means x is an even number. We only have $\exists x\, [P(x) \wedge Q(x)] \ \Rightarrow \ \exists x P(x) \ \wedge \ \exists x\, Q(x)$, which allows us to prove similarly [perhaps you ought to write it out] that $f(A \cap B) \subseteq f(A) \cap f(B)$, but in general $f(A \cap B) \neq f(A) \cap f(B)$.)[1]

[1] Dually, we have $\forall x\, P(x) \wedge Q(x) \Leftrightarrow \forall x\, P(x) \wedge \forall x Q(x)$, but only $\forall x\, P(x) \vee Q(x) \Leftarrow \forall x\, P(x) \vee \forall x Q(x)$. Give a counterexample to show the converse implication does not hold.

As an extra exercise, we prove here that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, where the inverse image is defined by $f^{-1}(A) = \{x : f(x) \in A\}$, the variable x having domain that of f.

Note that $x \in f^{-1}(A)$ if and only if $f(x) \in A$.

We have

$$\begin{aligned} x \in f^{-1}(A \cap B) \quad &\Leftrightarrow \quad f(x) \in A \cap B \\ &\Leftrightarrow \quad f(x) \in A \ \wedge \ f(x) \in B \\ &\Leftrightarrow \quad x \in f^{-1}(A) \ \wedge \ x \in f^{-1}(B) \\ &\Leftrightarrow \quad x \in f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

It is also the case that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, as can be seen by swapping $\vee$ for $\wedge$ (and $\cup$ for $\cap$) in the above.

(iii) Let $f(x) = x^2$ for $x \in \mathbb{R}$ and $A = \{x \in \mathbb{R} : -1 \leq x \leq 0\}$ and $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Show that $A \cap B = \{0\}$ and $f(A \cap B) = \{0\}$, while $f(A) = f(B) = \{y \in \mathbb{R} : 0 \leq y \leq 1\}$. Hence $f(A \cap B)$ is a proper subset of $f(A) \cap f(B)$.

Write down the sets $A \backslash B$ and $f(A) \backslash f(B)$ and show that it is *not* true that $f(A \backslash B) \subseteq f(A) \backslash f(B)$.

$A \cap B = \{0\}$ since $0 \in A \cap B$ and if $x \in A \cap B$ then $x \leq 0$ and $x \geq 0$, which together imply $x = 0$.

For $f(A \cap B)$ we have $f(\{0\}) = \{f(0)\} = \{0\}$.

By definition $f(A) = \{x^2 : -1 \leq x \leq 0\} = \{y : 0 \leq y \leq 1\}$ and $f(B) = \{x^2 : 0 \leq x \leq 1\} = \{y : 0 \leq y \leq 1\}$.

So here $f(A \cap B) = \{0\} \subsetneq [0, 1]$.

In interval notation, $A \setminus B = [-1, 0)$ and $f(A) \setminus f(B) = \emptyset$

6. Two sets A and B are *equinumerous* if there is a bijection $f : A \to B$. Show that the relation of being equinumerous is an equivalence relation.

Let $A \cong B$ denote the relation that A is equinumerous with B. Then $A \cong A$ since the identity map $f(x) = x$ gives a bijection from A to itself. Supposing $A \cong B$ it follows that $B \cong A$ since a bijection $f : A \to B$ has a bijective inverse $f^{-1} : B \to A$.

Finally, if $A \cong B$ and $B \cong C$ and $f : A \to B$ and $g : B \to C$ are two bijections exhibiting these equivalences, then the composite map $g \circ f : A \to C$ gives a bijection establishing $A \cong C$.

(I have assumed for this question that a bijection has a bijective inverse and that the composition of two bijections is again a bijection. To continue your soul's nutrition, you ought to prove these two facts.)

(i) For $a, b \in \mathbb{R}$ with $a < b$ give an explicit bijection from $A = \{x : a < x < b\}$ onto $B = \{y : 0 < y < 1\}$. Show that $\{x \in \mathbb{R} : x > 0\}$ is equinumerous with $\mathbb{R}$, and, finally, deduce that the set A is equinumerous with $\mathbb{R}$.

To map the interval (a, b) bijectively to $(0, 1)$ we translate by a leftwards to make $(0, b-a)$ and then scale by $\frac{1}{b-a}$ to obtain $(0, 1)$, i.e., the function $f(x) = \frac{x-a}{b-a}$ maps (a, b) bijectively to $(0, 1)$.

The interval $(0, \infty)$ is equinumerous with the whole real ine $(-\infty, \infty)$ by applying the bijection:

$$f(x) = \begin{cases} \frac{1}{x} - 1 & 0 < x \leq 1 \\ 1 - x & 1 < x. \end{cases}$$

This maps the subinterval $(0, 1]$ bijectively to $[0, \infty)$ and the subinterval $(1, \infty)$ bijectively to $(-\infty, 0)$.

Since $(0, 1)$ maps bijectively to $(0, \infty)$ via the map $f(x) = \frac{1}{x}$, we have, using the notation $\cong$ for equinumerous, $(a, b) \cong (0, 1) \cong (0, \infty) \cong (-\infty, \infty)$, and transitivity yields the desired result that $(a, b) \cong \mathbb{R}$.

(ii) A real number is *algebraic* if it is a solution of an equation of the form

$$a_0 + a_1x + a_2x^2 \cdots + a_nx^n = 0,$$

for some $n \in \mathbb{N}$ and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$.

Show that the set of algebraic numbers is equinumerous with $\mathbb{N}$. *(You may assume the fact that a set X is equinumerous with $\mathbb{N}$ if and only if there is a surjection from $\mathbb{N}$ onto X. Start with the fact that $\mathbb{Z}$ is equinumerous with $\mathbb{N}$ and go on to establish that there is a surjection from $\mathbb{N}$ onto the set of algebraic numbers.)*

This part of the question has been moved to Exercise Sheet 2 (questions 3 and 4(ii))