Mathematical Analysis I

Exercise sheet 11

7 January 2016

References: Abbott 6.6. Bartle & Sherbert 6.4

1. Let $f : [a, b] \to \mathbb{R}$ be such that f and its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b).

(i) Let $x_0 \in [a, b]$. Show that the polynomial $P_n(x)$ defined by

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

has the property that $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for each $k = 0, 1, \ldots, n$. [The polynomial P_n is called the *nth Taylor polynomial for f at x_0*.]

(ii) Taylor's Theorem with the Lagrange form for the remainder term states that, for any $x \in [a, b]$ there is $c \in (x_0, x)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1},$$

where P_n is the nth Taylor polynomial for f at x_0 defined in (i). Find the Taylor polynomial P_n for e^x at x_0 and show that the remainder term converges to 0 as $n \to \infty$ for each fixed x_0 and x. [*Use the fact that if (a_n) is a sequence of positive reals such that* $\lim a_{n+1}/a_n$ *exists and is* < 1 *then* $\lim a_n = 0$.]

(iii) Find the Taylor polynomial P_n for $f(x) = \sin x$ at $x_0 = 0$ and prove that the remainder term converges to 0 as $n \to \infty$ for each x.

(iv) Find the nth Taylor polynomial for $f(x) = (1 + x)^{-m}$ at $x_0 = 0$, where m is a positive integer.

2.

(i) Let $f, g : [a, b] \to \mathbb{R}$ be n times differentiable on (a, b). Use induction to prove Leibnitz's rule for the nth derivative of a product

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}(x),$$

for $x \in (a, b)$.

(ii) Let $h(x) = e^{-1/x^2}$ for $x \neq 0$ and $h(0) = 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does *not* converge to 0 as $n \to \infty$ for $x \neq 0$. [*By L'Hospital's Rule,* $\lim_{x\to 0} h(x)/x^k = 0$ *for any* $k \in \mathbb{N}$. *Use (i) to calculate* $h^{(n)}(x)$ *for* $x \neq 0$.]