Mathematical Analysis I

Exercise sheet 9

Selected solutions

17 December 2015

References: Abbott 5.2, 5.3, Bartle & Sherbert 6.2, 6.3

4.

(i) Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b) with $f'(x) \neq 1$. Suppose $f(x) = x$ and $f(y) = y$ for some $x, y \in [a, b]$. By the Mean Value Theorem on the interval $[x, y]$ we have

$$1 = \frac{x - y}{x - y} = \frac{f(x) - f(y)}{x - y} = f'(c)$$

for some $c \in (x, y)$. This contradicts the fact that $f'(c) \neq 1$ for $c \in (a, b)$. Hence there can be no $x \neq y$ that are both fixed points of f.

(ii) Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on $[a, b]$, with continuous derivative $f' : [a, b] \to \mathbb{R}$ and take $x < y \in [a, b]$. By the Mean Value Theorem

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

for some $c \in (x, y)$. Since f' is continuous on $[a, b]$, it is bounded and attains mininimum and maximum values (i.e., the image of $[a, b]$ under f' is a closed bounded interval). Hence there is a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$. Consequently, for any choice of $x, y \in [a, b]$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M,$$

which is to say that f is Lipschitz on the interval $[a, b]$.