Úlohy ke cvičení

Úloha 1: In the field $\mathbb{Z}_5$, solve the matrix equation

$$\begin{pmatrix} 1 & 4 & 2 & 2 & 2 & 3 \\ 4 & 2 & 0 & 0 & 4 & 2 \\ 4 & 2 & 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 0 & 4 & 4 \\ 3 & 0 & 2 & 2 & 4 & 4 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

Verify the result. You may use Sage, but then provide the commands and intermediate results.

$$(\mathbf{A}|\mathbf{B}) = \begin{pmatrix} 1 & 4 & 2 & 2 & 2 & 3 & | & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 & 4 & 2 & | & 3 & 2 & 3 \\ 4 & 2 & 0 & 1 & 1 & 1 & | & 0 & 3 & 4 \\ 0 & 2 & 4 & 0 & 4 & 4 & | & 0 & 0 & 2 \\ 3 & 0 & 2 & 2 & 4 & 4 & | & 0 & 1 & 4 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & 0 & 4 & 0 & 0 & 2 & | & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 & 2 & 2 & | & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 & | & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 \end{pmatrix}$$

$$\textit{Hence } \mathbf{X} = \begin{pmatrix} 2 + p_1 + 3p_3 & 3 + q_1 + 3q_3 & 4 + r_1 + 3r_3 \\ 3p_1 + 3p_2 + 3p_3 & 3q_1 + 3q_2 + 3q_3 & 1 + 3r_1 + 3r_2 + 3r_3 \\ p_1 & q_1 & r_1 \\ 2 + p_3 + 3p_2 & 1 + q_3 + 3q_2 & 1 + r_3 + 3r_2 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$$

Úloha 2: In the vector space $\mathbb{R}^4$ over the field $\mathbb{R}$ find the linear combination of vectors $(-5, 5, 1, -1)^T$, $(2, -5, 0, 2)^T$, $(3, 2, 0, -2)^T$ a $(2, -3, 1, 1)^T$ which does lead to vector $(-7, 12, 2, -4)^T$. Is this linear combination unique?

It is possible to find coefficients of of the linear combination with use of a system of equations. We may obtain for example $(2, 0, 1, 0)^T$ as a solution.

The system of equations does not have a unique solution, thus the linear combination is not unique. The general form of the solution is $(2, 0, 1, 0)^T + p(-1, -2, -1, 1)^T$.

Úloha 3: Let $\mathbf{A}$ be a matrix of size $m \times n$ over a field $\mathbb{K}$. Show that $Ker(\mathbf{A})$ forms a vector subspace in the arithmetic vector space $\mathbb{K}^n$.

Use the definition of a matrix kernel and show that the set is closed under summation and product with any element of the field $\mathbb{K}$.

Let $\mathbf{x}, \mathbf{x}' \in Ker(\mathbf{A})$, then from the definition $\mathbf{Ax} = \mathbf{Ax}' = \mathbf{0}$, then also $\mathbf{A}(\mathbf{x} + \mathbf{x}') = \mathbf{Ax} + \mathbf{Ax}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$, and so $\mathbf{x} + \mathbf{x}' \in Ker(\mathbf{A})$.

Similarly let $\mathbf{x} \in Ker(\mathbf{A})$ and $a \in \mathbb{K}$ then we compute $\mathbf{A}(a\mathbf{x}) = a(\mathbf{Ax}) = a\mathbf{0} = \mathbf{0}$, and so $a\mathbf{x} \in Ker(\mathbf{A})$.

Note that although there are three different multiplications in the first equation, it is possible to change the order in which the product with scalar will be done.

Úloha 4: Let $\mathbf{D}$ be a square matrix over a field $\mathbb{K}$. Show that all the matrices which commute in matrix product with matrix $\mathbf{D}$ form a vector space.

Show that it is a subspace of the vector space of all square matrices over the field $\mathbb{K}$.

All zero matrix commutes trivially with any matrix. Lets denote the set of all $\mathbf{D}$-commutable matrices as C.

Let $\mathbf{A}, \mathbf{B} \in C$, then $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{D} = \mathbf{AD} + \mathbf{BD} = \mathbf{DA} + \mathbf{DB} = \mathbf{D} \cdot (\mathbf{A} + \mathbf{B})$

Now let $\mathbf{A} \in C$ and $a \in \mathbb{K}$, then $(a\mathbf{A}) \cdot \mathbf{D} = a(\mathbf{A} \cdot \mathbf{D}) = a(\mathbf{D} \cdot \mathbf{A}) = \mathbf{D} \cdot (a\mathbf{A})$