Linear Algebra I

Exercise sheet 6

16/ 22 November 2016

1. Determine the powers p^{10} and q^{99} for the permutations $p = (5, 4, 3, 2, 1, 9, 8, 7, 6)$ and $q = (8, 6, 4, 2, 1, 3, 5, 7, 9)$.

2. Solve the permutation equation

$$p \circ x \circ q = \iota$$

(ι stands for the identity permutation) for x when $p = (1, 2, 7, 6, 5, 4, 3, 8, 9)$ and $q = (1, 3, 5, 7, 9, 8, 6, 4, 2)$.

3. Determine the sign of the permutation $(1, 4, 7, \ldots, 3n-2, 2, 5, 8, \ldots, 3n-1, 3, 6, \ldots, 3n)$.

4. For a permutation p of $[n]$ let $I(p) = \{i, j \in [n] : i < j, p(i) > p(j)\}$ denote the set of inversions of p. The sign of p is defined by $\mathrm{sgn}(p) = (-1)^{|I(p)|}$.

Give four different arguments to explain why $\mathrm{sgn}(p^{-1}) = \mathrm{sgn}(p)$.

[*For three of the arguments use different representations of a permutation: (1) by a bipartite graph in which arrows join i to $p(i)$ (the 2-line representation with arrows joining i in the top row to $p(i)$ in the bottom row), (2) by its cycle decomposition, and (3) as a product of transpositions. For the fourth, you may quote the identity* $\mathrm{sgn}(p \circ q) = \mathrm{sgn}(p)\mathrm{sgn}(q)$ *for permutations p and q — if feeling brave, try to prove this last identity too.*]