Linear Algebra I

Exercise sheet 4

1 November 2016

2.

$$\left[\begin{array}{cccc|c} a & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 \\ 1 & 1 & 1 & a & 1 \end{array}\right] \underset{\text{swap r1 and r4}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 1 & a & 1 & 1 & 1 \\ 1 & 1 & a & 1 & 1 \\ a & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\underset{\text{pivot c1}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & a-1 & 0 & 1-a & 0 \\ 0 & 0 & a-1 & 0 & 0 \\ 0 & 1-a & 1-a & 1-a^2 & 1-a \end{array}\right]$$

$$\underset{\text{r4}\leftarrow\text{r4+r2+r3}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & a-1 & 0 & 1-a & 0 \\ 0 & 0 & a-1 & 1-a & 0 \\ 0 & 0 & 0 & 2-a-a^2 & 1-a \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & a-1 & 0 & 1-a & 0 \\ 0 & 0 & a-1 & 1-a & 0 \\ 0 & 0 & 0 & (1-a)(a+3) & 1-a \end{array}\right]$$

When $a = 1$ the matrix is in echelon form as

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and the solution set is $\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 1\}$. Setting non-pivot variables x_2, x_3, x_4 as free parameters $p, q, r \in \mathbb{R}$ this is to say

$$\mathbf{x} = \begin{bmatrix} 1-p-q-r \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + p \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

When $a = -3$ the matrix takes the following echelon form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -3 & 1 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right],$$

for which there is no solution since the last row is inconsistent (all zero coefficients, non-zero in last column).

Otherwise, for $a \neq 1, -3$ we may reduce the ecehon form by scaling rows to obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{a+3} \end{array}\right].$$

We can solve the system of linear equations represented by this matrix by back-substitution: $x_4 = \frac{1}{a+3}$, $x_3 = x_4 = \frac{1}{a+3}$, $x_2 = x_4 = \frac{1}{a+3}$ and $x_1 = 1 - \frac{1}{a+3} - \frac{1}{a+3} - \frac{a}{a+3} = \frac{1}{a+3}$. Alternatively, we can put the matrix in reduced row echelon form to read of the solution vector in the last column:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{a+3} \end{array}\right] \underset{\text{r3}\leftarrow\text{r3+r4}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{a+3} \\ 0 & 0 & 0 & 1 & \frac{1}{a+3} \end{array}\right]$$

$$\underset{\text{r2}\leftarrow\text{ r2+r4}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & a & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{a+3} \\ 0 & 0 & 1 & 0 & \frac{1}{a+3} \\ 0 & 0 & 0 & 1 & \frac{1}{a+3} \end{array}\right]$$

$$\underset{\text{r1}\leftarrow\text{ r1-r2-r3-}a\text{r4}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{a+3} \\ 0 & 1 & 0 & 0 & \frac{1}{a+3} \\ 0 & 0 & 1 & 0 & \frac{1}{a+3} \\ 0 & 0 & 0 & 1 & \frac{1}{a+3} \end{array}\right]$$

from which there is the unique solution

$$\mathbf{x} = \begin{bmatrix} \frac{1}{a+3} \\ \frac{1}{a+3} \\ \frac{1}{a+3} \\ \frac{1}{a+3} \end{bmatrix} = \frac{1}{a+3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$