Úlohy ke cvičení

Úloha 1: In the vector space of real functions, determine whether the following polynomials are linearly independent or not.

$x^5+7x^3-9x^2+2x+3, \quad -x^5+x^4-5x^3+3x^2-6x-8, \quad -x^5+x^4-5x^3+4x^2-4x-4,$
$2x^4+4x^3-6x^2+4x+14$, and $3x^5-7x^4+7x^3+5x^2+14x+4$.

If they are linearly dependent, express some of them as a linear combination of others.

You may use Sage.

Úloha 2: Extend the set M to a basis of the vector space V

a) $M=\{(1,2,0,0)^T,(2,1,1,3)^T,(0,1,0,1)^T\}$, $V=\mathbb{R}^4$.

Choose a suitable basis that will contain candidates for the extension.

The space V is of dimension four, hence M should be extended by a single vector. From the exchange theorem follows that at least one vector of the canonical basis is independent on M. The independence could be resolved by solving $a_1u_1+a_2u_2+a_3u_3=e_i$, where u_1,u_2,u_3 are from M and e_i are vector of the canonical basis.

We get the matrix

$$\left(\begin{array}{ccc|cccc} 1&2&0&1&0&0&0\\ 2&1&1&0&1&0&0\\ 0&1&0&0&0&1&0\\ 0&3&1&0&0&0&1 \end{array}\right)\sim\cdots\sim\left(\begin{array}{ccc|cccc} 1&2&0&1&0&0&0\\ 0&1&0&0&0&1&0\\ 0&0&1&0&0&-3&1\\ 0&0&0&-2&1&6&-1 \end{array}\right)$$

Since the last row contains pivot in all columns of the right side, M could be extended by any vector of the canonical basis.

b) $M=\{-x^2,x+x^2,x^3-1\}$, in the space V of real polynomes of degree at most three.

We try to extend M by a vector from the basis $1,x,x^2,x^3$. We get the matrix

$$\left(\begin{array}{ccc|cccc} 0&0&-1&1&0&0&0\\ 0&1&0&0&1&0&0\\ -1&1&0&0&0&1&0\\ 0&0&1&0&0&0&1 \end{array}\right)\sim\cdots\sim\left(\begin{array}{ccc|cccc} -1&1&0&0&0&1&0\\ 0&1&0&0&1&0&0\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&0&1 \end{array}\right)$$

We see that either $M\cup\{1\}$ or $M\cup\{x^3\}$ (and many other possibilities which has not been tested) form a basis of V. On the other hand $M\cup\{x\}$ or $M\cup\{x^2\}$ do not form a basis.

Úloha 3: Determine dimensions and bases of the following subspaces of $\mathbb{Z}_5^7$.

a) $U_1=\mathcal{L}((4,1,0,3,4,0,0)^T,(4,3,1,0,2,3,1)^T,(4,1,4,0,3,2,4)^T,$
$(2,4,1,4,4,3,1)^T,(0,4,3,2,2,4,3)^T)$.

We build a matrix from the generators (vectors as rows) and we transform this matrix into the echelon form. The elementary transformations do no alter the row space, hence the resulting nonzero rows yield the desired basis.

$$\begin{pmatrix} 4&1&0&3&4&0&0\\ 4&3&1&0&2&3&1\\ 4&1&4&0&3&2&4\\ 2&4&1&4&4&3&1\\ 0&4&3&2&2&4&3 \end{pmatrix}\sim\sim\begin{pmatrix} 1&2&3&2&2&4&3\\ 0&1&1&0&2&3&1\\ 0&0&1&3&1&3&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$\dim(U_1)=3$.

A basis of U_1 is e.g. $(1,2,3,2,2,4,3)^T,(0,1,1,0,2,3,1)^T,(0,0,1,3,1,3,1)^T$.

b) $V_1=\{(x_1,\dots,x_7)^T\in\mathbb{Z}_5^7: x_1+3x_2+x_3+2x_4+3x_5+x_6+2x_7=0,$
$3x_1+4x_2+3x_3+x_4+4x_5+2x_6+4x_7=0,\ 2x_1+x_2+4x_3+4x_5+2x_7=0\}$.

We form the matrix of the system and find a basis of the solution space.

$$\begin{pmatrix} 1&3&1&2&3&1&2\\ 3&4&3&1&4&2&4\\ 2&1&4&0&4&0&2 \end{pmatrix}\sim\sim\begin{pmatrix} 1&3&1&2&3&1&2\\ 0&0&2&1&3&2&1\\ 0&0&0&0&0&4&3 \end{pmatrix}$$

The solution is $\mathbf{x}=p_1(2,1,0,0,0,0,0)^T+p_2(1,0,2,1,0,0,0)^T+p_3(1,0,1,0,1,0,0)^T+p_4(1,0,4,0,0,3,1)^T$.

The vectors by the parameters are the basis of the solution space, immediately could be seen that the dimension is the same as the number of free variables.

Úloha 4: Determine, whether the spaces U_i and V_i are in an inclusion. If so, find a basis of the larger one that extend a basis of the smaller one.

These subspaces of $\mathbb{Z}_5^7$ are defined as follows:

a) $U_1 = \mathcal{L}((4,1,0,3,4,0,0)^T, (4,3,1,0,2,3,1)^T, (4,1,4,0,3,2,4)^T,$
$(2,4,1,4,4,3,1)^T, (0,4,3,2,2,4,3)^T)$

$V_1 = \{(x_1,\ldots,x_7)^T \in \mathbb{Z}_5^7 : x_1 + 3x_2 + x_3 + 2x_4 + 3x_5 + x_6 + 2x_7 = 0,$
$3x_1 + 4x_2 + 3x_3 + x_4 + 4x_5 + 2x_6 + 4x_7 = 0,\ 2x_1 + x_2 + 4x_3 + 4x_5 + 2x_7 = 0\}$

The dimension of a subspace is smaller than the dimension of its superspace. Since we have calculated the dimensions earlier in one of the previous problems, we may exclude the case $V_1 \subset U_1$. It remains to verify whether the space are in the opposite inclusion or whether they are incomparable. It suffices to verify whether $\dim(\mathcal{L}(U_1 \cup V_1)) = \dim(V_1) = 4$.

Alternatively we may try to express the vectors of the basis of the smaller space as a linear combination of the vectors from the other basis (this is in principle the same).

				2	1	0	0	0	0	0
				1	0	2	1	0	0	0
				1	0	1	0	1	0	0
				1	0	4	0	0	3	1
2	2	2	3	1	2	3	2	2	4	3
1	0	2	1	0	1	1	0	2	3	1
0	3	1	1	0	0	1	3	1	3	1

The rows of the first matrix are the coordinates of the vectores of the basis of U_1 with respect to the basis V_1 (both bases have been calculated earlier). Observe thet these coordinates na be straightforwardly determined from the 2., 4., 5. and 7. component of the vector, argue why.

To extend the basis we start with any basis of the smaller space and insert vectors of the larger until we get dimension 4.

It holds that $U_1 \subset V_1$.

This inclusion can be seen in an easier way: all vectors that generate U_1 satisfy equations of the definition of V_1.