Immersion and embedding of 2-regular digraphs

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1 Introduction

In this section we will be interested in 2-regular digraphs (i.e. digraphs for which every vertex has both indegree and outdegree equal to 2). There is a natural operation called splitting which takes a 2-regular digraph and reduces to a new 2-regular digraph. To *split* a vertex v with inward edges uv and $u'v$ and outward edges vw and vw', we delete the vertex v and then add either the edges uw and $u'w'$, or we add the edges uw' and $u'w$. If G, H are 2-regular digraphs, we say that H is *immersed* in G if a graph isomorphic to H may be obtained from G by a sequence of splits.

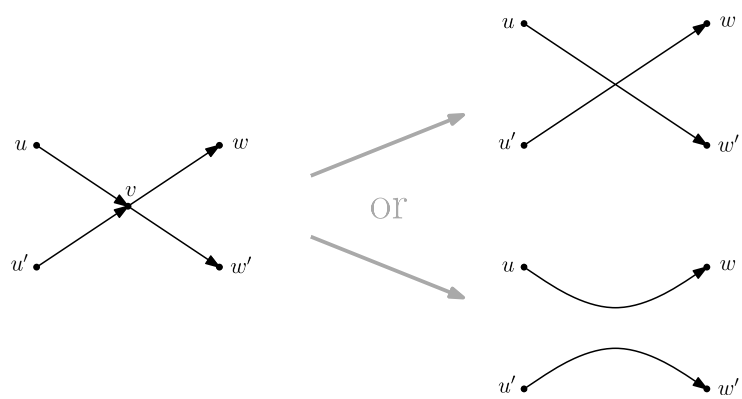

Figure 1: splitting a vertex

Our central goal in this article will be to show how the theory of 2-regular digraphs under immersion behaves similar to the theory of (undirected) graphs under graph minor operations. We will begin with some motivation. Consider an ordinary undirected graph G which is embedded in an orientable surface. The *medial* graph H is constructed from G by the following procedure. For every edge e of G add a vertex of the graph H in the centre of e. Now, whenever two edges $e, f \in E(G)$ are consecutive at a vertex (or equivalently, consecutive along a face) we add an edge between the corresponding vertices of H. Based on this construction, every vertex of the original graph is contained in the centre of a face of the medial graph, and every face of the original graph completely contains a new face of the medial graph. So, the faces of the medial graph have a natural bipartition into these two types, and indeed this gives a proper 2-face colouring of our medial graph. Since our surface is orientable, we may direct the edges so that every face containing a vertex of the original graph is oriented clockwise. The following figure gives an example of this oriented medial graph.

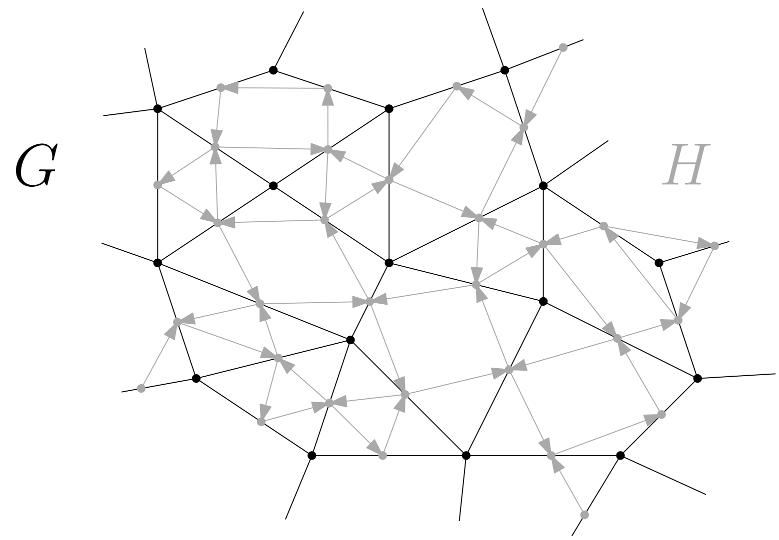

Figure 2: A graph G and its oriented medial H

Let us first note that this oriented medial graph is a 2-regular digraph. Now let's consider how the medial graph H changes when we delete an edge e of the original graph. Suppose that v is the vertex of the medial graph which corresponds to e. After deleting e the new medial graph will no longer have the vertex v, and (check this!) in fact, the new oriented medial may be obtained from the original by splitting v. Similarly, if we modify the original graph by contracting e, the new medial may be obtained by doing the other split at v. So, in this setting, we see that our minor operations on G correspond precisely to splitting vertices of the oriented medial. This connection suggests a general study of 2-regular digraphs under immersion, and this will be our direction going forward.

There is a key feature of the embedded 2-regular digraphs coming from the aforementioned construction. Namely, at each vertex v in this embedding, the cyclic order of the incident edges goes inward-outward-inward-outward.

As you can easily see, if this is the local behaviour at v, then either of the possible ways of splitting v will result in a new 2-regular digraph which still has a natural embedding in the surface. Motivated by this, let us now define a *special* embedding of a 2-regular digraph to be one which satisfies this property at every vertex. Now, similar to the behaviour of (undirected) graphs under minor operations, we have the following easy observation for our 2-regular digraphs.

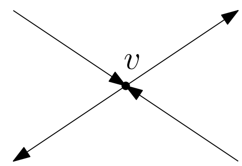

Figure 3: a nice local rotation

Observation 1.1. *If G is a 2-regular digraph embedded in a surface $\mathcal{S}$, then every digraph immersed in G also embeds in $\mathcal{S}$.*

The Graph Minors project of Robertson and Seymour established a number of very deep properties of (undirected) graphs under the relation of minors. One great achievement of this project is a rough structure theorem for the class of graphs not containing a fixed graph H as a minor. A remarkable consequence of this is that every proper minor closed class of graphs (ex. planar graphs) is characterized by a finite list of excluded minors (i.e minor minimal graphs not in the class). A Ph.D. student of Seymour named Johnson proved an analogous rough structure theorem for 2-regular digraphs under immersion (which strongly features special embeddings). He claims to know a proof of the finite list of excluded immersed graphs, but this was never written.

One very pleasing property of 2-regular digraphs is that their behaviour under immersion is somewhat cleaner and simpler than that of usual graphs under minors. As evidence for this, we offer the following chart which gives information about the number of minor minimal graphs not embeddable in certain surfaces, and the analogous list of immersion-minimal graphs with no special embedding. This theorem for the plane will be given in the next section and I'm unclear who deserves credit for it (probably either Johnson or Seymour). The projective plane theorem for 2-regular digraphs is due to Archdeacon, D., Hannie, and Mohar. The same group expects to have the corresponding result for the torus shortly, and I have optimistically filled this entry.

Surface	Minors (graphs)	Immersion (2-reg. digraphs)
plane	$K_{3,3}$, K_5 (Kuratowski, Wagner)	$C_3^{(2)}$
proj. plane	35 graphs (Archdeacon)	$C_3^{(2)} + C_3^{(2)}$, $C_3^{(2)} \cdot C_3^{(2)}$, $C_4^{(2)}$, C_6^2
torus	$> 10^4$ graphs, unsolved	hopefully coming soon!

To explain our notation here, let us assume G and G' are digraphs. Then $G^{(2)}$ is the digraph obtained from G by adding a new edge in parallel with each existing edge, and G^2 is the digraph obtained by adding a new edge from u to v whenever there is a directed path of length 2 from u to v. The disjoint union of G and H is denoted $G+G'$ and we let $G \cdot G'$ denote a digraph obtained from the disjoint union of G and G' by choosing edges $(u, v) \in E(G)$ and $(u', v') \in E(G')$, deleting them and then adding the edges (u, v') and (u', v). Finally, we let C_k denote a directed cycle of length k.

2 Planar Embeddings

Our goal in this section is to prove the following result.

Theorem 2.1. *A 2-regular digraph has a special embedding in the plane if and only if it does not immerse* $C_3^{(2)}$*.*

Proof. First we prove the "only if" direction. By Observation 1.1, it suffices to show that $C_3^{(2)}$ has no special embedding in the plane. To see this, first note that in a special embedding, every face is bounded by a closed directed walk. Since these directed walks must use every edge exactly twice, every special embedding of $C_3^{(2)}$ has at most 4 faces. So, by Euler's formula, if we have a special embedding of $C_3^{(2)}$, the Euler characteristic of the surface must be at most $3 - 6 + 4 = 1$.

For the "if" direction, we may assume that our digraph G is connected. Choose an Euler tour W of G, let $v \in V(G)$ and consider the behaviour of the tour W at v. The tour W must pass through v twice, say using the edges (u, v) then (v, w) and later using the edges (u', v) then (v, w'). Now modify the graph G to by uncontracting a new (undirected) edge at the vertex v forming the two adjacent vertices v, v' so that we now have the directed edges (u, v), (v, w) and (u', v'), (v', w').

If we do this at every such vertex, we obtain a mixed graph (with both directed and undirected edges) which we call H. The graph H has a directed cycle containing every vertex as given by our original Euler tour. We shall view H as drawn with this cycle as a circle and all other edges as chords. So, we will think of each vertex of the original graph as a chord of this circle.

Based on this figure, we now construct a new graph K with vertex set $V(G)$ and an edge between u, v if the chords corresponding to u and v cross. This type of graph is known as a *circle graph*. We now split into cases depending on whether K is bipartite. If K is a bipartite graph, then we may partition our chords into two sets $\{A, B\}$ so that no two in the same set cross. Based on this we can embed the graph H in the plane by putting the chords in A on the inside and the chords in B outside of the circle. Once we have an embedding of H, we can contract all of these chords to obtain a special embedding of our original graph G.

The remaining possibility is that K is not bipartite, and in this case we may choose an induced odd cycle $C \subseteq K$. For every vertex v of the original graph which is not in $V(C)$ split the vertex v in accordance with the Euler tour (i.e. so if the edges (u, v) and (v, w) appear consecutively in the tour, we split v to add the edge (u, w)). Let G' be the 2-regular digraph obtained by doing this for every vertex not in $V(C)$, and let W' be the Euler tour obtained from W. Using the same process as before, we let H' be the mixed graph obtained from G' by uncontracting, and let K' be the corresponding circle graph. Observe that by this operation, the resulting graph K' is precisely C.

If our cycle $C = K'$ has length > 3 then we will modify it to make it shorter by two. To do this, we simply choose two consecutive vertices on our cycle and split them in the original graph G' in a manner not in accordance with our Euler tour W'. The reader may verify that the resulting 2-regular digraph, say G'' will have an associated circle graph K'' which is still a cycle but is now two vertices shorter. By repeating this process, we may obtain a 2-regular graph G^* immersed in G with the property that the circle graph K^* associated with G^* is a triangle. It follows that G^* is the digraph $C_3^{(2)}$, as desired. □

3 Peripheral Cycles

Although our result for the projective plane isn't terribly complicated, it does require some preliminary lemmas, most of which are quite sensible and meaningful. In this section we will sketch a proof of one of these tools.

For an undirected graph G, we say that a cycle C is *peripheral* if there is no edge $e \in E(G) \setminus E(C)$ with both ends in $V(C)$, and the graph $G - V(C)$ is connected. If G is embedded in the plane, then it is easy to see that every peripheral cycle must be the boundary of a face.

Theorem 3.1 (Tutte)**.** *If G is a 3-connected graph, then every edge is in at least two peripheral cycles.*

Corollary 3.2. *A 3-connected planar graph has a unique embedding in the plane.*

We will prove an analogous theorem for 2-regular digraphs. In such a digraph G, we define a directed cycle C to be *peripheral* if $G - E(C)$ is strongly connected. If G is any 2-regular graph which has a special embedding in the plane, then deleting the edges of any directed cycle separates the part of the graph inside this cycle from the outside. So, as before, in this case any peripheral cycle must be a face boundary. Our goal here will be to prove the following.

Theorem 3.3. *Every strongly 2-edge-connected 2-regular digraph has at least two peripheral cycles through every edge.*

Corollary 3.4. *Every strongly 2-edge-connected 2-regular digraph which has a special embedding in the plane has a unique special embedding in the plane.*

Proof of Theorem 3.3. Let $e = (u, v)$ be an edge of G. Our first goal will be to find one peripheral cycle through e. To do this, we choose a directed path P from v to u so as to lexicographically maximize the sizes of the components of $G' = G - (E(P) \cup \{e\})$. That is, we choose the path P so that the largest component of G' is as large as possible, and subject to this the second largest is as large as possible, and so on.

Suppose (for a contradiction) that G' has components $G_1, G_2, \ldots, G_k$ with $k > 1$ where G_k is a smallest component. Let P' be the minimal directed path of P which contains all vertices of G_k and suppose the start of P' is the vertex x and the last vertex is y. By construction, G_k must contain both x and y. Furthermore, since G_k is Eulerian, both of these vertices have indegree and outdegree equal to one in G_k. If there is a component G_i with $i < k$ which contains a vertex in the interior of P', then we may choose a directed path P'' in G_k from x to y (since G_k is Eulerian, it is automatically strongly connected). Now we get a contradiction, since we can reroute the original path along P'' instead of P' and get a new path which improves our lexicographic ordering. Thus, all vertices in the interior of P' must also be in G_k. However, in this case $G_k \cup P'$ is an induced subgraph which is separated from the rest of the graph by just two edges, and we have a contradiction to the strong 2-edge-connectivity. It follows that $k = 1$, so the cycle $P \cup \{e\}$ is indeed peripheral.

Since the cycle $P \cup \{e\}$ is peripheral, there exists a directed path Q with $E(Q) \cap E(P) = \emptyset$ from v to u. Among all such directed paths Q we choose one so that the unique component of $G - (E(Q) \cup \{e\})$ which contains P is as large as possible, and subject to that we lexicographically maximize the sizes of the remaining components. By the same argument as above, this choice will result in another peripheral cycle. $\square$