Flows in bidirected graphs

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1 Colouring-flow duality in the plane

We begin with a lovely observation due to Tutte which opened the study of this field. Before stating it we will need to introduce some basic terminology.

Definition: Let Γ be an abelian group (written additively), and let $G = (V, E)$ be a directed graph. We define a function $\phi : E \to \Gamma$ to be a *flow* if the following condition (called the Kirchoff rule) is satisfied at every vertex $v \in V$

$$\sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e) = 0.$$

So, in words, a function is a flow, if at every vertex v, the sum of the values on the incoming edges is equal to the sum of the values on the outgoing edges. We say that a flow is a *k-flow* when $\Gamma = \mathbb{Z}$ and $|\phi(e)| < k$ for every $e \in E$; we call ϕ *nowhere-zero* if $\phi(e) \neq 0$ for every $e \in E$. Note that if we have a flow, then we can reverse an edge and change its value to $-\phi(e)$ and this preserves the Kirchoff rule, so we still have a flow. This also preserves the properties of k-flow and nowhere-zero flow. Accordingly, we will say that an undirected graph has a *nowhere-zero* Γ *flow* or *nowehere-zero k-flow* if some (and thus every) orientation of it has this property.

Theorem 1.1 (Tutte)**.** *If G and G^* are dual planar graphs, then G^* has a proper k-colouring if and only if G has a nowhere-zero k-flow.*

Proof of the "only if" direction. (see next section for the "if" direction) Let V^* be the set of vertices of G^* and also the set of faces of G and suppose that $g : V^* \to \{0, 1, \ldots, k-1\}$ is a proper k-colouring. Now, orient the edges of G arbitrarily and assign each edge e of G a value $\phi(e)$ according to the rule that $\phi(e) = g(a) - g(b)$ where a is the face to the left of the directed edge e (when it is oriented upward) and b is the face to the right. To check that this is a flow, consider a vertex v and suppose first that all edges are directed away from v. In this case, the Kirchoff rule will be satisfied at v because within this sum each face a incident with v contributes $g(a) - g(a) = 0$. If we flip the direction of an edge, this flips its sign, so the Kirchoff rule will still be satisfied. Since our colouring was proper, the resulting function ϕ is indeed a nowhere-zero k-flow, as desired. $\square$

Note that a planar graph with a loop edge does not have any proper colouring, so it's dual does not have any nowhere-zero k-flow. More generally, any graph with a cut-edge will not have a nowhere-zero Γ-flow for any (abelian) group Γ. To see this, just sum the Kirchoff rule over all vertices in one component of $G - e$ for a cut-edge e. Since we have a flow, this sum must be zero, but all terms in this sum apart from $\phi(e)$ cancel, so it gives $\phi(e) = 0$. Based on the above theorem connecting flows and colourings, Tutte made three remarkable conjectures concerning the existence of nowhere-zero flows, all of which are still open despite considerable efforts.

Conjecture 1.2 (Tutte). *Let G be a graph without a cut-edge.*

1. *Then G has a nowhere-zero 5-flow.*
2. *If G has no Petersen minor, it has a nowhere-zero 4-flow.*
3. *If G is 4-edge-connected, it has a nowhere-zero 3-flow.*

The first of these conjectures, the 5-flow conjecture, holds true for planar graphs by the 5-colour theorem. The Petersen graph does not have a nowhere-zero 4-flow, so if it is true, the 5-flow conjecture would be best possible. Seymour proved that every graph without a cut-edge has a nowhere-zero 6-flow, and this result will be of significance for our forthcoming discussion.

The 4-flow conjecture when restricted to cubic graphs is equivalent to the statement that every cubic graph with no cut-edge and no Petersen minor has a 3-edge colouring. This was proved by Robertson, Sanders, Seymour and Thomas, but little more is known in general. The last of these conjectures holds true for planar graphs since it dualizes to the statement that every triangle free planar graph is 3-colourable—which was proved by Grötzsch.

Before leaving this section let us close with another easy observation and another useful theorem of Tutte. Observe that our proof of the "only if" direction of Theorem 1.1 actually gives a somewhat more general result. If instead of choosing a k-colouring using the colours $\{0, 1, \dots, k-1\}$ we had instead chosen Γ to be an abelian group of order k and chosen $g : E \to \Gamma$ to be our colouring, then the construction would have resulted in a nowhere-zero Γ flow. So, a k-colouring of the dual naturally gives us either a nowhere-zero k-flow or a nowhere-zero Γ-flow in the original graph G. The following theorem shows that this phenomena holds true in a more general setting.

Theorem 1.3 (Tutte). *For every positive integer k and graph G, the following are equivalent.*

1. *G has a nowhere-zero k-flow.*
2. *G has a nowhere-zero Γ flow for some group Γ with $|\Gamma| = k$.*
3. *G has a nowhere-zero Γ flow for every group Γ with $|\Gamma| = k$.*

The utility of this result becomes immediately apparent when one starts working with flows. The reason is that it is easy to modify a Γ-flow to get another Γ-flow (for instance by adding a constant value to all edges on a directed cycle), but it is generally difficult to modify a k-flow to get another k-flow.

2 Duality for orientable surfaces

Let's consider a directed graph $G = (V, E)$ which is embedded in an orientable surface. Let $\phi : E \to \Gamma$ be a flow on G. Now we may construct the dual graph $G^* = (V^*, E^*)$ and orient its edges so that whenever $e \in E$ corresponds to $e^* \in E^*$, the edge e^* crosses left to right over e. Now consider the function $\phi^* : E^* \to \Gamma$ given by the rule $\phi^*(e^*) = \phi(e)$. For a walk W in G^* with edge sequence $e_1^*, \ldots, e_m^*$ we define the *height* of this walk to be

$$\phi^*(W) = \sum_{i=1}^{m} \epsilon_i \phi(e_i^*)$$

where $\epsilon_i = 1$ if e_i^* is traversed forward, and $\epsilon_i = -1$ if e_i^* is traversed backward. With this notation in place, we see that the Kirchoff rule for a vertex v in the original graph is precisely equivalent to the statement that the closed walk bounding the face of G^* corresponding to v has height 0. So, our flow ϕ dualizes to give a function ϕ^* with the property that every facial walk has height 0. This is an important concept, so let's pause to define it.

Definition: For an embedded directed graph G and a function $\psi : E(G) \to \Gamma$, we say that ψ is a *local-tension* if the height of every facial walk is 0. We say that ψ is a *tension* if every closed walk has height 0.

Just as with flows, we will call a (local) tension ψ *nowhere-zero* if $\psi(e) \neq 0$ for every $e \in E(G)$ and we call ψ a k-*(local) tension* if $\Gamma = \mathbb{Z}$ and $|\psi(e)| < k$ for every $e \in E(G)$. Also just like flows, we can reverse the direction of an edge and multiply its value by -1 to obtain a new (local) tension, so the question of when a directed graph has a nowhere-zero (local) tension depends only on the underlying graph and not the orientation. The following key result shows that nowhere-zero tensions are essentially the same as colourings.

Proposition 2.1. *A graph G has a nowhere-zero Γ-tension if and only if it has a proper $|\Gamma|$-colouring.*

Sketch of proof. For the "if" direction, choose a Γ-colouring of G given by $g : V(G) \to \Gamma$. Then orient the edges of G arbitrarily and assign the value $\psi(e) = g(v) - g(u)$ if e is an edge directed from v to u. It is straightforward to check that this gives a nowhere-zero tension.

For the"only if" direction choose a nowhere-zero tension $\psi : E(G) \to \Gamma$ and then fix a base point $u \in V(G)$. Now for every vertex $v \in V(G)$ choose a walk W_v from u to v and define $g(v) = \psi(W_v)$. It follows from the assumption that ψ is a tension that the value $g(v)$ does not depend on the choice of W_v. Moreover, the assumption that ψ was nowhere-zero means that the resulting function g is a proper colouring. □

Assume that we have a tension ψ of an embedded graph G, and assume that every face in this embedding is a disc. If W is a closed walk in the graph which forms a contractible curve in the surface, then we may deform W to a trivial walk by rerouting along faces one at a time. It follows from this that $\psi(W) = 0$. More generally, let us fix a base point $u \in V(G)$ and consider two closed walks W and W' starting and ending at u. If W and W' are homotopic, then by the same argument, we deduce that $\psi(W) = \psi(W')$. This leads us to the following key property.

Proposition 2.2. *Let G be a directed graph embedded in a surface $\mathcal{S}$. If $\psi : E(G) \to \Gamma$ is a local-tension, then ψ induces a group homomorphism from $\pi_1(S) \to \Gamma$. This homomorphism is trivial if and only if ψ is a tension.*

With this, we can now return to prove the other part of our first theorem.

Proof of "if" direction of Theorem 1.1. By assumption, the graph G has a nowhere-zero k-flow. So, by Theorem 1.3 we may orient G and choose a nowhere-zero $\mathbb{Z}_k$ flow ϕ. Let G^* be the oriented dual (as above) and define $\phi^*(e^*) = \phi(e)$ for every edge $e^* \in E(G^*)$. Then ϕ^* is a nowhere-zero $\mathbb{Z}_k$-local tension. However, since the homotopy group of the plane is trivial, the above proposition implies that ϕ^* is actually a tension. Thus by Proposition 2.1 the graph G^* has a proper k-colouring. □

3 Duality for nonorientable surfaces

Now we shall start off with a directed graph G which is embedded in a non-orientable surface, and a local tension $\psi : E(G) \to \Gamma$. Our aim is to translate the local-tension property into a kind of flow in the dual graph. However, since our surface is not orientable, there is no obvious orientation of the dual to use. In fact, we will need a more complicated notion. A *bidirected graph* is a graph in which every edge has two arrowheads, one associated with each endpoint. Just as with usual directed graphs, these arrowheads may be directed either toward or away from this endpoint.

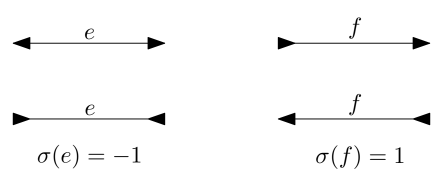

Figure 1: edge types

We assume (as usual) that every face of the embedded graph G is a disc, and we will think of each of these discs as equipped with a local notion of clockwise. (This is one of the many ways of working with nonorientable surface embeddings.) Let G^* be the dual graph, and consider the face of G which is associated to some vertex $v^* \in V$. We have chosen a clockwise orientation of this face, and we let W_{v^*} be a facial walk which traverses this face clockwise. Now, by assumption we have $\psi(W_{v^*}) = 0$ and we shall translate this into a flow type condition at the vertex v^* in the dual. To do so, just mark each edge e^* of G^* which is incident with v^* with an arrowhead directed to v^* if the corresponding edge e of G is forward in W_{v^*} and with an arrowhead the opposite direction if it is backward. This immediately translates the property $\psi(W_{v^*}) = 0$ into the Kirchoff rule being satisfied at v^*. However, if we do this at every vertex of the dual, we will in general end up with a bidirected orientation of this dual graph

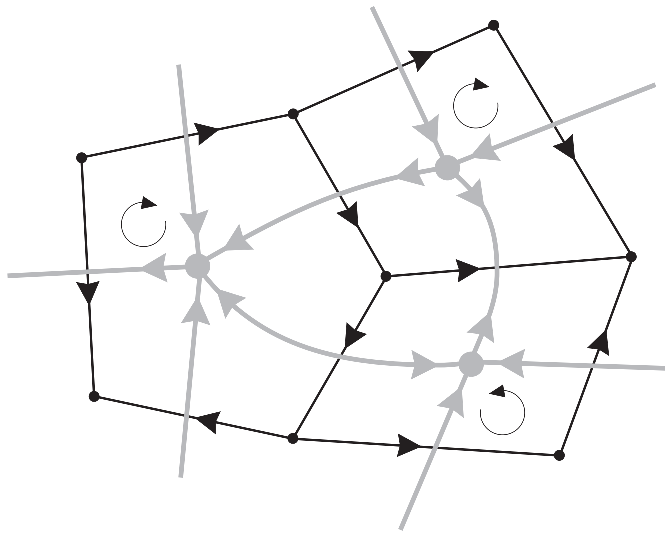

Figure 2: duality

Following the above process and giving the dual graph G^* a bidirected orientation results in the duality we want. Namely, we will have that our local tension ψ of G translates into a flow ψ^* of the dual (bidirected) graph G^*. So, just as we could use nowhere-zero flows in ordinary digraphs to construct local-tensions on orientable surfaces, we can now use nowhere-zero flows in bidirected graphs to construct local-tension on non-orientable surfaces. One might have hoped that the analogue of Tutte's 5-flow conjecture would still hold true for bidirected graphs, that is that every bidirected graph without the obvious obstruction has a nowhere-zero 5-flow, but this is not true. To see why, consider the dual graphs K_6 and Petersen embedded in the projective plane. Direct the edges of K_6 and use the above procedure to give Petersen a bidirected orientation. Now consider any local tension $\phi : E(K_6) \to \mathbb{Z}$ of K_6. By Proposition 2.2 this local tension induces a group homomorphism from the fundamental group of our surface, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ to the group $\mathbb{Z}$. Since this must be trivial, we deduce that ϕ must be a tension. It follows that this embedded K_6 does not have a 5-local-tension, and then by duality the associated bidirected Petersen does not have a nowhere-zero 5-flow. Bouchet conjectured that this was the most extreme example.

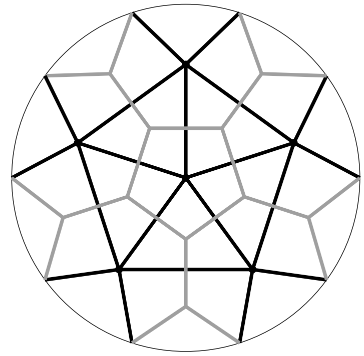

Figure 3: Petersen and K_6 in the projective plane

Conjecture 3.1 (Bouchet)**.** *Every bidirected graph with a nowhere-zero $\mathbb{Z}$-flow has a nowhere-zero 6-flow.*

Bouchet proved that graphs with nowhere-zero $\mathbb{Z}$-flows have nowhere-zero 216-flows. This was improved to 30-flows by Fouquet and independently by Zyka. We have shown that such graphs have nowhere-zero 12-flows.