Discrete Mathematics

Exercise sheet 9

28 November/ 6 December 2016

1. How many graphs on the vertex set $[2n] = \{1, 2, \ldots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set $\{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$?

[*Hint: any such graph arises by pairing off the* $2n$ *vertices that are to be joined by edges. The number of ways to do this can be counted as follows: choose which vertex to pair with vertex* 1 *(*$2n-1$ *choices). This leaves* $2n-2$ *vertices to pair off. Repeat this procedure by taking the smallest of the remaining vertices and deciding which one it will pair off with. At each step you have a number of free choices; multiply these together to find how many ways there are in total.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 4.1.]

2. Let G be a graph with adjacency matrix A_G. Show that G contains a triangle (i.e. a copy of K_3) if and only if there exist indices i and j such that both the matrices A_G and A_G^2 have a nonzero entry in the (i,j)-position.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 4.2.]

3. Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 4.3.]

4. Let T be a tree with n vertices, $n \geq 2$. For a positive integer i, let p_i be the number of vertices of T of degree i.

(a) Prove that
$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} = 2.$$

[*Use the fact that* T *has* n *vertices and* $n-1$ *edges to write down two linear relations between the* p_i*. Note also that* $p_i = 0$ *for* $i \geq n$*.*]

(b) Deduce from (a) the end-vertex lemma, that a tree with at least two vertices has at least two end-vertices.

(c) Deduce from (a) that a tree with a vertex of degree k has at least k vertices of degree 1.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 5.1.]