Discrete Mathematics

Exercise sheet 6

7/ 11 November 2016

1. An opinion poll reports that the percentage of voters who would be satisfied with each of three candidates Rumpty, Conlint and Peabrain for US President is 65%, 57%, 58% respectively. Further, 28% would accept Rumpty or Conlint, 30% Rumpty or Peabrain, 27% Conlint or Peabrain, and 12% would be happy with any of the three. Use the principle of inclusion-exclusion to assess the veracity of this statement: what do you conclude?

[*Hint: according to these figures, what is the percentage of voters who reject all three candidates?*]

Let R, C, P denote the sets of voters satisfied with Rumpty, Conlint and Peabrain respectively. Let S be the set of all voters (so $R \cup C \cup P \subseteq S$).

By PIE,

$$|S \setminus (R \cup C \cup P)| = |\overline{R} \cap \overline{C} \cap \overline{P}| = |S| - |R| - |C| - |P| + |R \cap C| + |R \cap P| + |C \cap P| - |R \cap C \cap P|$$

Divide this equation through by $|S|$ and multiply through by 100 in order to obtain the following expression for the percentage of voters who reject all candidates:

$$100\% - 65\% - 57\% - 58\% + 28\% + 30\% + 27\% - 12\%$$

but this gives -7%, which is impossible (a proportion must be nonnegative). So there must be some mistake in the figures.

2.

(a) (Sieve of Eratosthenes) How many numbers are left from the set $\{1, 2, 3, \ldots, 1000\}$ after all multiples of $2, 3, 5$ and 7 are crossed out?

Let $S = \{1, 2, \ldots, 1000\}$ and let A_d denote the subset of elements that are multiples of d. We wish to find $|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}|$.

First, we observe that $|A_d| = \lfloor \frac{1000}{d} \rfloor$ (because $\lfloor \frac{1000}{d} \rfloor d \leq 1000$ while $(\lfloor \frac{1000}{d} \rfloor + 1)d > 1000$).

Thus $|A_2| = 500$, $|A_3| = 333$, $|A_5| = 200$, $|A_7| = 142$.

Since $2, 3, 5, 7$ are distinct primes we have $A_2 \cap A_3 = A_6$ and likewise for the other 5 intersections of two sets ($\binom{4}{2} = 6$) and $A_2 \cap A_3 \cap A_5 = A_{30}$ and likewise for the other 3 intersections of three sets ($\binom{4}{3} = 4$). Finally, $A_2 \cap A_3 \cap A_5 \cap A_7 = A_{210}$.

Then by PIE

$$\begin{aligned}
|\overline{A_2} \cap \overline{A_3} \cap \overline{A_5} \cap \overline{A_7}| &= |S| - |A_2| - |A_3| - |A_5| - |A_7| + |A_6| + |A_{10}| + |A_{14}| + |A_{15}| + |A_{21}| + |A_{35}| \\
&\quad - |A_{30}| - |A_{42}| - |A_{70}| - |A_{105}| + |A_{210}| \\
&= 1000 - 500 - 333 - 200 - 142 + 166 + 100 + 71 + 66 + 47 + 28 \\
&\quad - 33 - 23 - 14 - 9 + 4 \\
&= 228.
\end{aligned}$$

Thus there are 228 numbers between 1 and 1000 that are not multiples of 2,3,5 or 7.

In the sieve of Eratosthenes numbers are striked out when they are a multiple of a strictly smaller number – so the primes 2,3,5,7 are not removed, leaving 228+4= 232 numbers not striked out.

(b) How many numbers $n < 1000$ are not divisible by the square of any integer greater than 1 (such numbers are called *square-free*, for example $7, 15, 21$ are square-free, but 9 and 12 are not).

A number less than 1000 is divisible by a square of an integer greater than 1 if and only if it is divisible by one of the squared primes $4, 9, 25, 49, 121, 169, 289, 361, 529, 841, 961 (= 31^2)$ (because if divisible by a square s^2 then divisible by the square of a prime factor of s, and we finish the list at 31^2 since $32^2 = 1024 \geq 1000$ so the next prime squared 37^2 is larger than 1000).

We may take $S = \{1, 2, \ldots, 1000\}$ (because 1000 is not squarefree so does not contribute to the count) in order to use some of calculations in (a).

As in (a), let A_d denote the set of numbers in S that are divisible by d. For relatively prime c, d we have $A_c \cap A_d = A_{cd}$.

For $d > 1000$ we have $A_d = \emptyset$. In applying PIE we may therefore miss out terms $|A_d|$ in which $d > 1000$.

We then obtain

$$|\overline{A}_4 \cap \overline{A}_9 \cap \overline{A}_{25} \cap \overline{A}_{49} \cap \overline{A}_{121} \cap \overline{A}_{169} \cap \overline{A}_{289} \cap \overline{A}_{361} \cap \overline{A}_{529} \cap \overline{A}_{841} \cap \overline{A}_{961}| =$$

$$\begin{aligned}&1000 - |A_4| - |A_9| - |A_{25}| - |A_{49}| - |A_{121}| - |A_{169}| - |A_{289}| - |A_{361}| - |A_{529}| - |A_{841}| - |A_{961}|\\ &+ |A_{36}| + |A_{100}| + |A_{196}| + |A_{484}| + |A_{676}| + |A_{225}| + |A_{441}|\\ &- |A_{900}|\end{aligned}$$

This gives

$$\begin{aligned}&1000 - \lfloor\frac{1000}{4}\rfloor - \lfloor\frac{1000}{9}\rfloor - \lfloor\frac{1000}{25}\rfloor - \lfloor\frac{1000}{49}\rfloor - \lfloor\frac{1000}{121}\rfloor - \lfloor\frac{1000}{169}\rfloor - \lfloor\frac{1000}{289}\rfloor - \lfloor\frac{1000}{361}\rfloor - \lfloor\frac{1000}{529}\rfloor - \lfloor\frac{1000}{841}\rfloor - \lfloor\frac{1000}{961}\rfloor\\ &+ \lfloor\frac{1000}{36}\rfloor + \lfloor\frac{1000}{100}\rfloor + \lfloor\frac{1000}{196}\rfloor + \lfloor\frac{1000}{484}\rfloor + \lfloor\frac{1000}{676}\rfloor + \lfloor\frac{1000}{225}\rfloor + \lfloor\frac{1000}{441}\rfloor\\ &- \lfloor\frac{1000}{900}\rfloor\end{aligned}$$

which is equal to

$$1000 - 250 - 111 - 40 - 20 - 8 - 5 - 3 - 2 - 1 - 1 - 1 + 27 + 10 + 5 + 2 + 1 + 4 + 2 - 1 = 608.$$

[Let $Q(n)$ denote the number of squarefree integers $< n$. So $Q(1000) = 608$ by the above. For $n = 10, 100, 1000, 10000, \ldots$ the value of $Q(n)$ is $7, 61, 608, 6083, 60794,\ 607926, 6079291,$ $60792694, 607927124,$6079270942$, \ldots$ The ratio $Q(n)/n$ appears to approach $0 \cdot 607927$.... In fact, the limit of $Q(n)/n$ as $n \to \infty$ does exist and is equal to $6/\pi^2$.]

3. Recall from lectures the formula

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

for the number of derangements of $[n]$ (permutations of $[n]$ with no fixed point).

[*You do not need to prove it here.*]

(a) Determine the number of permutations of $[n]$ with exactly one fixed point.

A permutation has exactly one fixed point if and only if it is a derangment of $n-1$ elements (fixing one element from $[n]$). There are n choices for which element is fixed and $D(n-1)$ derangments of the remaining elements. Hence there are $nD(n-1)$ such permutations.

(b) For $0 \le k \le n$, determine the number of permutations of $[n]$ with exactly k fixed points.

[*Hint: first choose the k points to be fixed. The remainder of the points are not fixed, i.e., the permutation on these remaining $n-k$ elements is a derangement.*]

There are $\binom{n}{k}$ choices of k elements from $[n]$ that are fixed. The permutation is then a derangement of the remaining $n-k$ elements, of which there are $D(n-k)$.

Hence there are $\binom{n}{k}D(n-k)$ permutations that fix exactly k elements.

As an additional remark – not required by the question – we use the above formula for $D(n)$ to find that the number of permutations fixing exactly k elements is given by

$$\begin{aligned}\binom{n}{k}D(n-k) &= \binom{n}{k}(n-k)!\sum_{i=0}^{n-k}(-1)^i\frac{1}{i!}\\ &= \frac{n!}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-k}\frac{1}{(n-k)!}\right)\end{aligned}$$

(c) Deduce from (b) the formula

$$D(n) = n! - nD(n-1) - \binom{n}{2}D(n-2) - \cdots - \binom{n}{n-1}D(1) - 1.$$

Partition the set of $n!$ permutations according to the number k of fixed points it has. By (b) we have

$$n! = 1 + \sum_{k=0}^{n-1}\binom{n}{k}D(n-k),$$

in which we have put the case of n fixed points out of the sum (there is just one permutation fixing all points, namely the identity permutation).[1] This give the desired formula.

Note that $D(1) = 0$, corresponding to the fact that a permutation fixing $n-1$ elements must fixed the remaining element as well, so there are no permutations of $[n]$ that fix exactly $n-1$ points.

[1]Alternatively, we set $D(0) = 1$.