Discrete Mathematics

Exercise sheet 6

7/ 11 November 2016

1. An opinion poll reports that the percentage of voters who would be satisfied with each of three candidates Rumpty, Conlint and Peabrain for US President is 65%, 57%, 58% respectively. Further, 28% would accept Rumpty or Conlint, 30% Rumpty or Peabrain, 27% Conlint or Peabrain, and 12% would be happy with any of the three. Use the principle of inclusion-exclusion to assess the veracity of this statement: what do you conclude?

[*Hint: according to these figures, what is the percentage of voters who reject all three candidates?*]

2.

(a) (Sieve of Eratosthenes) How many numbers are left from the set $\{1, 2, 3, \ldots, 1000\}$ after all multiples of $2, 3, 5$ and 7 are crossed out?

(b) How many numbers $n < 1000$ are not divisible by the square of any integer greater than 1 (such numbers are called *square-free*, for example $7, 15, 21$ are square-free, but 9 and 12 are not).

3. Recall from lectures the formula

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

for the number of derangements of $[n]$ (permutations of $[n]$ with no fixed point).

[*You do not need to prove it here.*]

(a) Determine the number of permutations of $[n]$ with exactly one fixed point.

(b) For $0 \leq k \leq n$, determine the number of permutations of $[n]$ with exactly k fixed points.

[*Hint: first choose the k points to be fixed. The remainder of the points are not fixed, i.e., the permutation on these remaining $n - k$ elements is a derangement.*]

(c) Deduce from (b) the formula

$$D(n) = n! - nD(n-1) - \binom{n}{2} D(n-2) - \cdots - \binom{n}{n-1} D(1) - 1.$$