Discrete Mathematics

Exercise sheet 5

31 October/ 3 November 2016

1. Let $P = (X, \preceq)$ be a partially ordered set.
   Let $x \prec y$ denote $[x \preceq y \ \wedge \ x \neq y]$.

   (a) Define what is meant for $a \in X$ to be a *minimal element* of P, and for it to be *minimum element* of P.

   a is *minimal*:
   $$\neg[\exists x \in X \quad x \prec a]$$
   (there is no element below a), which is logically equivalent to
   $$\forall x \in X \quad \neg(x \prec a)$$
   (every element is not below a). Alternatively
   $$\forall x \in X \quad [x \preceq a \ \Rightarrow \ x = a].$$
   For a partial order $\neg(x \prec a)$ is not in general the same as $x \succeq a$. (However, for a total order this is true, amounting to the law of *trichotomy*: $x \prec a$ or $x = a$ or $x \succ a$.)

   a is *minimum*:
   $$\forall x \in X \quad a \preceq x.$$

   (b) Likewise, define the terms *maximal element* and *maximum element*.

   a is *maximal*:
   $$\neg[\exists x \in X \quad a \prec x]$$
   (there is no element above a), which is logically equivalent to
   $$\forall x \in X \quad \neg(a \prec x).$$
   Alternatively
   $$\forall x \in X \quad [a \preceq x \ \Rightarrow \ x = a].$$

   a is *maximum*:
   $$\forall x \in X \quad x \preceq a.$$

   (c) Show that a maximum element is always maximal.

   a is maximum ($\forall x \in X \quad x \preceq a$) implies a is maximal ($\forall x \in X \quad \neg(a \prec x)$) because $x \preceq a$ implies $\neg(a \prec x)$ (by antisymmetry of $\preceq$).

   [The converse does not hold since as already remarked $\neg(a \prec x)$ does not necessarily imply $x \preceq a$: a maximal element need not be maximum.]

(d) Give an example of a partially ordered set with a maximal element but no maximum element.

Examples include $(X, |)$ for $X \subseteq \mathbb{N}$ containing an incomparable pair of numbers (i.e. neither a divisor of the other) whose product is not in X (say, $\{1, 2, 3\}$). Also, the Booelan poset on a set X with the maximum X removed, i.e., $(2^X \backslash \{X\}, \subseteq)$, the set of all proper subsets of X. Another example is an antichain on X for $|X| \geq 2$.

(e) Find an example of a partially ordered set which has a maximum element, but which has no minimum element and no minimal element.

Examples include $(\mathbb{Z} \setminus \mathbb{N}, \leq)$ or the real interval $(a, b]$, which has maximum element b but no minimal elements (the infimum a to this subset of $\mathbb{R}$ lies outside the set: for every $c \in (a, b]$ there is $x \in (a, c)$).

Both these examples are total orders. The following is a partial order with a maximum element but having no minimal elements and which is not a total order:

Define X to be the set of all finite strings on $\{N, E\}$ (imagine these as representing a walk on the square grid, in which N is a step one unit north (up) and E a step one unit east (right)). The empty string is also counted (as the walk that does not move from the starting point).

Define $x \preceq y$ when the string y appears as the initial string of x (so x begins with the walk y and then continues some further steps).

Then $(X, \preceq)$ is a partial order, with maximum element the empty string (every walk starts) and no minimal elements (for any walk x there is a walk that continues it further).

[A *rooted tree* T with set of nodes X defines a partial order with $x \preceq y$ when y lies on the path from the root to x (the root of T is the maximum element). If a is a leaf of T then a is a minimal element of this poset: there is no node below a leaf, so a lies on no path from the root other than that which ends at itself.

The poset just defined is equivalent to that defined using the infinite binary tree (a node is a walk x, each node x has two descendants xN and xE, according as the walk continues north or east). The root is the maximum element, and there are no minimal elements (leaves).]

2. Let $P = (X, \preceq)$ be a finite partially ordered set.

(a) Define what is meant by a *linear extension* of P. [*See* Matoušek & Nešetřil, Invitation to Discrete Mathematics, *section 2.2*]

A *linear extension* of $P = (X, \preceq)$ is a total order $\leq$ on X such that

$$\forall x, y \in X \quad [x \preceq y \Rightarrow x \leq y]$$

Equivalently, a linear extension of P is a bijection $f : X \to [n]$, where $n = |X|$, such that

$$\forall x, y \in X \quad [x \preceq y \Rightarrow f(x) \leq f(y)]$$

In terms of the Hasse diagram of P, the function f is a labelling of the elements of X by integers in $[n]$ with the property that labels increase up each chain of P.

(b) Prove that P has at most $n!$ linear extensions, where $n = |X|$. Which posets P have exactly $n!$ linear extensions?

A linear extension of P is a total order on its element set X. There are $n!$ total orders on a set of n elements (total orders correspond to permutations – see Exercise Sheet 5, question 2(e)).

Therefore there are at most $n!$ linear extensions.

If $x \preceq y$ in $P = (X, \preceq)$ then a linear extension of P must have $x \leq y$, which implies the number of total orders extending P is at most $n!/2$ (because to every total order on X with $x < y$ there corresponds a total order on X with $x > y$, for example by taking the reverse of the given total order, which is again a total order).

Hence for there to be $n!$ linear extensions of P, no two elements of P can be comparable, i.e. P is an antichain of n elements. In this case, any total order of the elements of P is a linear extension (the required condition $x \preceq y \Rightarrow x \leq y$ is vacuously true: the hypothesis is never satisfied).

(c) Show that if P has just one linear extension then P is a total order, and, conversely, if P is a total order then it has just one linear extension.

In place of the total order $P = (X, \preceq)$, we shall work with the strict total order $(X, \prec)$, in which $x \prec y$ if and only if $x \preceq y$ and $x \neq y$.

First we prove the statement: $P = (X, \prec)$ is a total order $\Rightarrow$ $P = (X, \prec)$ has a unique linear extension $(X, <)$.

For any pair of distinct elements $x, y \in X$ we have $x \prec y$ or $y \prec x$ (since $\prec$ defines a total order), but not both (by antisymmetry).

Suppose $<$ and $<'$ are two distinct linear extensions of P. Then there exist $x, y \in X$ such that $x < y$ and $y <' x$ (since $<$ and $<'$ are not the same linear order).

But if $x \prec y$ then $x < y$ and $x <' y$ (since both $<$ and $<'$ are linear extensions of P). But this contradicts the assumption that $y <' x$. Similarly, if $y \prec x$ then $y < x$ and $y <' x$. This contradicts the assumption that $x < y$. Hence our supposition that there are two distinct linear extensions of P must be false. So P has at exactly one linear extension (given by taking $<$ the same as $\prec$).

Now we prove the statement: $P = (X, \prec)$ has a unique linear extension $(X, <) \Rightarrow P = (X, \prec)$ is a total order

Let us prove the contrapositive statement: $P = (X, \prec)$ is not a total order $\Rightarrow$ $P = (X, \prec)$ has more than one linear extension $(X, <)$.

By the assumption that $(X, \prec)$ is not a total order, we have two distinct incomparable elements $a, b \in X$ (neither $a \prec b$ nor $b \prec a$).

Suppose $a < b$ in a given linear extension $(X, <)$ of P. We wish to construct a linear extension $(X, <')$ of P in which $b <' a$ (this will then give us at least two distinct linear extensions of P).

Consider the relation $\prec'$ on X defined by

$$x \prec' y \quad \text{if } x \prec y, \text{ or } x \preceq b \text{ and } a \preceq y$$

(In the Hasse diagram of P we are adding a line joining b up to a – we might have to shift a up higher than b, but can do so whilst preserving all the other lines.)

The relation $\prec'$ is a partial order on X (obvious from the Hasse diagram interpretation) and a linear extension of $(X, \prec')$ is a linear extension of $(X, \prec)$. This is because a linear extension $<'$ of $(X, \prec')$ in particular satisfies $x <' y$ if $x \prec y$, plus the further condition that $b <' a$, and conditions on any elements comparable with b or a.

Any such linear extension $(X, <')$ of $(X, \prec')$ provides a linear extension of $P = (X, \prec)$ in which $b <' a$, which is different to the linear extension $(X, <)$ of P, for which by assumption $a < b$. $\square$

3. Let $P = (X, \preceq)$ be a partially ordered set.

(a) Define what is meant by a *chain* of P and an *antichain* of P.

A *chain* in a poset $P = (X, \preceq)$ is a subset $Y \subseteq X$ such that the restriction of P to Y is a total order, i.e., for each $y_1, y_2 \in Y$ either $y_1 \preceq y_2$ or $y_2 \preceq y_1$.

An *antichain* in a poset $P = (X, \preceq)$ is a subset $Z \subseteq X$ such that the restriction of P to Z is a poset in which no two elements are comparable, i.e., for each $z_1, z_2 \in Z$ if $z_1 \preceq z_2$ or $z_2 \preceq z_1$ then $z_1 = z_2$.

(b) Let X be a finite set with $|X| = n$. Show that the length of the longest chain in the partially ordered set $(2^X, \subseteq)$ is $n + 1$.

Let $(2^X, \subset)$ be the strict partial order derived from $(2^X, \subseteq)$ in which $A \subset B$ means $A \subseteq B$ and $A \neq B$ (A is a proper subset of B). A chain in $(2^X, \subset)$ is a chain in $(2^X, \subseteq)$, and conversely.

Let $A_1 \subset A_2 \subset \cdots \subset A_k$ be a chain in $(2^X, \subset)$.

Then $0 \leq |A_1| < |A_2| < \cdots < |A_k| \leq n$. Since the sizes of sets are integers we can say more: $|A_i| + 1 \leq |A_{i+1}|$ for $i = 1, \ldots, k - 1$, from which, $k - 1 \leq |A_1| + k - 1 \leq |A_k| \leq n$.

It follows that $k \leq n + 1$.

An example of a chain $A_1 \subset A_2 \subset \cdots \subset A_{n+1}$ is provided by, for example, $A_1 = \emptyset$ and $A_i = [i - 1]$ for $i = 2, 3, \ldots, n + 1$.

Hence $\omega((2^X, \subseteq)) = n + 1$.

[*The size of the largest antichain in* $(2^X, \subseteq)$ *is more challenging to find: look up* Sperner's theorem *for the answer.*]