Discrete Mathematics

Exercise sheet 5

31 October/ 3 November 2016

1. Let $P = (X, \preceq)$ be a partially ordered set.

   (a) Define what is meant for $a \in X$ to be a *minimal element* of P, and for it to be *minimum element* of P.

   (b) Likewise, define the terms *maximal element* and *maximum element.*

   (c) Show that a maximum element is always maximal.

   (d) Give an example of a partially ordered set with a maximal element but no maximum element.

   (e) Find an example of a partially ordered set which has a maximum element, but which has no minimum element and no minimal element.

2. Let $P = (X, \preceq)$ be a finite partially ordered set.

   (a) Define what is meant by a *linear extension* of P. [*See* Matoušek & Nešetřil, Invitation to Discrete Mathematics*, section 2.2*]

   (b) Prove that P has at most $n!$ linear extensions, where $n = |X|$. Which posets P have exactly $n!$ linear extensions?

   (c) Show that if P has just one linear extension then P is a total order, and, conversely, if P is a total order then it has just one linear extension.

3. Let $P = (X, \preceq)$ be a partially ordered set.

   (a) Define what is meant by a *chain* of P and an *antichain* of P.

   (b) Let X be a finite set with $|X| = n$. Show that the length of the longest chain in the partially ordered set $(2^X, \subseteq)$ is $n + 1$.

   [*The size of the largest antichain in* $(2^X, \subseteq)$ *is more challenging to find: look up* Sperner's theorem *for the answer.*]