Discrete Mathematics

Exercise sheet 4

24 October/ 1 November 2016

1. [*Bookwork*] Let $R \subseteq X \times X$ be a relation on a set X. Define what is means for R to be

   (a) reflexive,

   (b) symmetric,

   (c) anti-symmetric,

   (d) transitive,

   (e) an equivalence relation,

   (f) a partial order,

   (g) a linear order.

2. The *adjacency matrix* of a binary relation R on $[n] = \{1, 2, \ldots, n\}$ is the matrix whose (i, j)-entry is defined for $i, j \in [n]$ by

$$a_{i,j} = \begin{cases} 1 & (i,j) \in R \\ 0 & (i,j) \notin R. \end{cases}$$

(See Section 1.5 of Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, for a detailed exposition.)

   (a) How many relations are there on $[n]$ in total? [*Hint: an $n \times n$ matrix with entries* 0 *or* 1 *defines the adjacency matrix of a relation. Count how many such matrices there are.*]

   (b) How many reflexive relations are there on $[n]$?

   (c) How many symmetric relations are there on $[n]$?

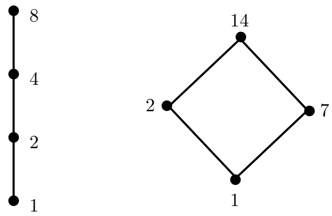
   (d) How many anti-symmetric relations are there on $[n]$? [*Hint: for a pair (i, i) there are two choices (either $(i, i) \in R$ or $(i, i) \notin R$), while for (i, j) with $i \neq j$ there are three mutually exclusive choices, $(i, j) \in R$, $(j, i) \in R$ or neither.*]

   (e) How many linear orders are there on $[n]$? [*You may find the adjacency matrix point of view not so helpful to answer this question, but rather take another viewpoint.*]

3. Let D_n be the set of divisors of n. Show that the relation $\preceq$ on D_n defined by $a \preceq b$ if and only if a divides b is a partial order.

(b) For $n = 2, 3, \ldots, 11$ draw the Hasse diagram of the poset $(D_n, \preceq)$ of divisors of n.

For example, the posets of divisors of 8 and 14 are as below:



(c) What property does the number n have if $(D_n, \preceq)$ is a linear order (as for $n = 8$)?

(d) When is $(D_n, \preceq)$ isomorphic to the poset $([m], \subseteq)$ for some m (as is the case for $n = 14$ with $m = 2$)?

(e) What is the size of the longest chain in $(D_n, \preceq)$?

What is the size of the largest antichain in $(D_n, \preceq)$? [*Hint: give your answer in terms of the factorization of n into a product of prime powers. A* prime power *is a number of the form p^a for some prime p and integer $a \geq 1$. For a number $n > 1$ we have $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_m^{a_m}$ for primes $p_1, \ldots, p_m$ and integers $a_1, \ldots, a_m \geq 1$. For the above examples, $8 = 2^3$ and $14 = 2 \cdot 7$.*]