Discrete Mathematics

Exercise sheet 2

10 /13 October 2016

2. Let $f : X \to Y$ and $g : Y \to X$ be functions such that $(g \circ f)(x) = x$ for each $x \in X$ and $(f \circ g)(y) = y$ for each $y \in Y$. Prove that f and g are bijections.

[Note that in the orginal question $g : Y \to Z$, but we must have $Z = X$.]

First we establish that f must be an injection. Suppose that $f(x_1) = f(x_2)$. Then $x_1 = (g \circ f)(x_1) = (g \circ f)(x_2) = x_2$.

Second we establish that f must be a surjection.

We are given that for each y we have $y = (f \circ g)(y) = f(g(y))$. Also, $g(y) = x$ for some $x \in X$ (the range of g is X). Thus $y = f(x)$. In other words, to each $y \in Y$ there is some $x \in X$ such that $f(x) = y$, i.e., f is onto.

A similar argument shows that g must be a bijection.

[The functions f and g are inverse to each other.]

3.

(a) Let A be a set. What is the set $A \times \emptyset$ equal to?

The definition of the Cartesian product of two sets A and B is

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

If $B = \emptyset$ then there is no element $b \in B$, hence $A \times \emptyset = \emptyset$.

(b) Let A, B, C be sets. Under what conditions does it follow from $A \times C = B \times C$ that $A = B$?

Given that

$$A \times C = \{(a, c) : a \in A, c \in C\} = \{(b, c) : b \in B, c \in C\} = B \times C,$$

if $C \neq \emptyset$ then choose $c \in C$ and we must have

$$A \times \{c\} = \{(a, c) : a \in A\} = \{(b, c) : b \in B\} = B \times \{c\},$$

because $(a, c) = (b, c')$ if and only if $a = b$ and $c = c'$. By projecting on to the first coordinate, i.e. by the bijection $(x, c) \mapsto x$ we conclude that $A = B$.

4. Let X be a finite set and let 2^X denote the set of all subsets of X.

(a) Prove that $|2^X| = 2^{|X|}$.

(1) Proof by induction. The empty set has just 1 subset (itself) and $2^0 = 1$. (A singleton set has two subsets, the empty set and itself, and $2^1 = 2$.)

Suppose the assertion is true for $|X| = n$. (We have verified this is the case for $n = 0$ and $n = 1$.)

Let Y be a set of $n+1$ elements. Write $Y = X \cup \{y\}$ where $|X| = n$ and $y \notin X$.

A subset S of Y either contains y or does not contain y. In the first case $S = T \cup \{y\}$ for a subset T of X, and in the second case S is subset of X. Conversely, for each subset T of X, the set $T \cup \{y\}$ is a subset of Y, and each subset of X is a subset of Y.

Hence $|2^Y| = |2^X| + |2^X| = 2 \cdot 2^{|X|} = 2^{n+1} = 2^{|Y|}$.

This completes the induction step and the proof.

(2) Proof by binary indicator vector. Let $X = \{x_1, \ldots, x_n\}$ and encode subsets S of X by a binary sequence $(e_1, e_2, \ldots, e_n)$ where

$$e_i = \begin{cases} 1 & x_i \in S \\ 0 & x_i \notin S \end{cases}$$

There is a bijection between binary sequences $(e_1, \ldots, e_n)$ and subsets of X, and the number of binary sequences of length n is 2^n. (Actually, to prove this obvious statmenent formally requires a similar inductive argument.) An equivalent formulation is to encode subsets of X by functions $f : X \to \{0, 1\}$.

(b) Prove that $2^X = 2^Y$ if and only if $X = Y$.

If $X = Y$ then $2^X = 2^Y$ is clear.

For the converse, suppose $2^X = 2^Y$ and for a contradiction that $X \neq Y$. Without loss of generality we may assume there exists $x \in X$ such that $x \notin Y$. (Otherwise $X \subset Y$ and swap the roles of X and Y in this proof.)

Then $\{x\} \in 2^X$ but $\{x\} \notin 2^Y$, whence $2^X \neq 2^Y$. This is the desired contradiction, hence we must have $X = Y$.

Alternative proof: use the fact that for any set X we have $X = \bigcup\{S : S \subseteq X\} = \bigcup 2^X$. Given $2^X = 2^Y$, taking unions we have $X = Y$.

5. Describe the relation $R \circ R$ if R stands for

For $R \subseteq X \times X$, the composition $R \circ R$ is the relation defined by $(x, z) \in R \circ R$ if and only if there exists $y \in X$ such that $(x, y) \in R$ and $(y, z) \in R$.

(a) the equality relation "=" on the set $\mathbb{N}$ of natural numbers,

$(x, z) \in R \circ R$ iff there is y such that $x = y$ and $y = z$. This is the case iff $x = z$. Hence $R \circ R = R$ in this case.

(b) the relation "less than or equal to" ("$\leq$") on $\mathbb{N}$,

$(x, z) \in R \circ R$ iff there is y such that $x \leq y$ and $y \leq z$. If $x \leq y$ and $y \leq z$ then $x \leq z$ by transitivity of $\leq$. Conversely, if $x \leq z$ then there exists such y (e.g. take $y = x$ or $y = z$). Hence, $R \circ R = R$ is $\leq$.

(c) the relation "strictly less than" ("$<$") on $\mathbb{N}$,

If $x < y$ and $y < z$ then $x < z$ by transitivity of $<$.

If $x < z - 1$ then there exists such y (e.g. take $y = x + 1 < z$). However, if $x \geq z - 1$ then there is no such y, as necessarily $y \geq x + 1 \geq z$. Hence, $R \circ R$ is defined by $(x, z) \in R \circ R$ if and only if $x + 1 < z$.

(d) the relation "strictly less than" ("$<$") on the set $\mathbb{R}$ of real numbers. If $x < y$ and $y < z$ then $x < z$ by transitivity of $<$.

Conversely, if $x < z$ then there exists such a y (e.g. take $y = \frac{1}{2}(x + z) < z$). Hence, $R \circ R = R$ is $<$.