Discrete Mathematics

Exercise sheet 1

3 /6 October 2016

1. Prove the following statements by mathematical induction:

(a) $\sum_{i=1}^{n}(2i-1) = n^2$.

Base case: true for $n = 1$, as $1 = 1^2$.

Induction hypothesis: assume true for given $n \geq 1$, i.e.

$$\sum_{i=1}^{n}(2i-1) = n^2. \tag{1}$$

Then

$$\begin{aligned}\sum_{i=1}^{n+1}(2i-1) &= \sum_{i=1}^{n}(2i-1) + (n+1) \\ &\underset{\text{by (1)}}{=} n^2 + (2n+1) \\ &= (n+1)^2,\end{aligned}$$

which is the statement for $n + 1$.

By induction the statement is true for all integers $n \geq 1$.

(b) $6n^2 + 2n$ is a multiple of 4.

Base case: true for $n = 0$ as 0 is a multiple of 4 (and for $n = 1$ as 8 is a multiple of 4).

Induction hypothesis: assume true for given $n \geq 0$, i.e.

$$6n^2 + 2n = 4m \quad \text{for some } m \in \mathbb{N} \tag{2}$$

Then

$$\begin{aligned}6(n+1)^2 + 2(n+1) &= 6n^2 + 12n + 6 + 2n + 2 \\ &= (6n^2 + 2n) + 12n + 8 \\ &\underset{\text{by (2)}}{=} 4m + 4(3n+2) \\ &= 4(m + 3n + 2),\end{aligned}$$

which is to say that the statement holds for $n + 1$.

By induction the statement is true for all integers $n \geq 0$.

(c) $\prod_{i=2}^{n} \frac{i-1}{i} = \frac{1}{n}$.

Base case: true for $n = 2$ as $\frac{2-1}{2} = \frac{1}{2}$.

Induction hypothesis: assume true for given $n \geq 2$, i.e.

$$\prod_{i=2}^{n} \frac{i-1}{i} = \frac{1}{n}. \tag{3}$$

Then

$$\begin{aligned}
\prod_{i=2}^{n+1} \frac{i-1}{i} &= \left(\prod_{i=2}^{n} \frac{i-1}{i}\right) \cdot \frac{n}{n+1} \\
&\underset{\text{by (3)}}{=} \frac{1}{n} \cdot \frac{n}{n+1} \\
&= \frac{1}{n+1},
\end{aligned}$$

which is to say the statement holds for $n + 1$.

By induction the statement is true for all integers $n \geq 2$.

(d) $\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

Base case: true for $n = 1$ as $1^2 = 1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6}$.

Induction hypothesis: assume true for given $n \geq 1$, i.e.

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \tag{4}$$

in which we have made the factorization

$$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1).$$

(Made in retrospect when carrying out the induction step below, where it makes calculation easier.)

Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&\underset{\text{by (4)}}{=} \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\
&= \frac{1}{6}(n+1)[n(2n+1) + 6(n+1)] \\
&= \frac{1}{6}(n+1)[2n^2 + 7n + 6)] \\
&= \frac{1}{6}(n+1)[(n+2)(2n+3)] \\
&= \frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1)
\end{aligned}$$

which is to say the statement holds for $n + 1$.

By induction the statement is true for all integers $n \geq 1$.

Try to give an alternative proof for each of (a), (b) and (c) that does not use induction.

(a) Twice the given sum is

$$\begin{array}{ccccccc} 1 & + & 3 & + & 5 & + \cdots + (2n-1) & + \\ (2n-1) & & + (2n-3) + \cdots + 3 + & & 1 & & \end{array}$$

and pairing off terms $1+(2n-1) = 2n$, $3+2n-3 = 2n$, ... , $(2n-1)+1 = 2n$ we have n times $2n$, i.e. $2n^2$ in total. Halving this gives $1+3+\cdots+(2n-1) = n^2$. More formally,

$$\begin{aligned} 2\sum_{i=1}^{n}(2i-1) &= \sum_{i=1}^{n}(2i-1) + \sum_{i=1}^{n}(2n-(2i-1)) \\ &= \sum_{i=1}^{n}[(2i-1)+2n-(2i-1)] \\ &= \sum_{i=1}^{n} 2n \\ &= 2n^2, \end{aligned}$$

from which $\sum_{i=1}^{n}(2i-1) = \frac{1}{2}(2n^2) = n^2$.

(b) If $n = 2m$ is even then $6n^2+2n = 24m^2+4m = 4(6m^2+m)$ and if $n = 2m+1$ is odd then

$$\begin{aligned} 6n^2+2n &= (24m^2+24m+6)+4m+2 \\ &= 24m^2+28m+8 \\ &= 4(6m^2+7m+2) \end{aligned}$$

(c) This one can be seen by cancelling numerator and denominator of successive terms in the product, i.e.,

$$\begin{aligned} \prod_{i=2}^{n}\frac{i-1}{i} &= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n} \\ &= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdots\frac{\cancel{n-2}}{\cancel{n-1}}\cdot\frac{\cancel{n-1}}{n} \\ &= \frac{1}{n}. \end{aligned}$$

However, this is not really different to the induction proof as you need to use the etcetera (dot dot dot ...) to stand in for arbitrarily many factors to which you apply cancellation. Here the induction proof formalizes the intuitive argument of cancelling $n-2$ denominator-numerator pairs in the product of $n-1$ factors.

2. Prove the following statements using the method of proof by contradiction:

(a) There is no largest natural number.

Suppose that N is a natural number larger than all other natural numbers. Then $N+1$ is a natural number, and $N+1 > N$, a contradiction. Hence no such largest natural number N exists.

(b) If n^2 is an odd number then n is odd.

Suppose for a contradiction that n^2 is odd and n is even. Then $n^2 + n = n(n+1)$ is even (as it is divisible by n) and odd (as it is the sum of an odd numebr and even number), which is impossible. Hence if n^2 is odd then n cannot be even, i.e. n is odd.

(c) If a, b and c are natural numbers such that $a^2 + b^2 = c^2$ then either a is even or b is even.
Suppose for a contradiction that $a^2 + b^2 = c^2$ while neither a nor b is even, i.e. both are odd. Say $a = 2u + 1$ and $b = 2v + 1$. Then

$$\begin{aligned} a^2 + b^2 &= (4u^2 + 4u + 1) + (4v + 4v + 1) \\ &4(u^2 + u + v^2 + v) + 2 \end{aligned}$$

so that c^2 is even. Therefore c is even (suppose not, then $c = 2w + 1$ is odd and $c^2 = 4(w^2 + w) + 1$ is odd, a contradiction). But if $c = 2d$ is even, then $c^2 = 4d^2$ is a multiple of 4, while $a^2 + b^2 = 4(u^2 + u + v^2 + v) + 2$ is not a multiple of 4. This contradiction implies our supposition that a and b are both odd is false. Hence at least one of a and b is even.

3. Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and for $n \geq 1$ by the recurrence $F_{n+1} = F_n + F_{n-1}$. Prove the following statements by mathematical induction:

(a) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Base case: for $n = 0$, $F_0 = 0 = 1 - 1 = F_2 - 1$.

Induction hypothesis: assume true for given $n \geq 0$, i.e. $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Then

$$\begin{aligned} \sum_{i=0}^{n+1} F_i &= F_{n+1} + \sum_{i=0}^{n} F_i \\ &\underset{\text{by ind. hyp.}}{=} F_{n+1} + F_{n+2} - 1 \\ &= F_{n+3} - 1 = F_{(n+1)+2} - 1, \end{aligned}$$

which is to say the statement holds for $n + 1$. By induction the statement is true for all integers $n \geq 0$.

(b) $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.

Base case: for $n = 0$, $F_0^2 = 0 = 0 \cdot 1 = F_0 F_1$.

Induction hypothesis: assume true for given $n \geq 0$, i.e. $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.

Then

$$\begin{aligned} \sum_{i=0}^{n+1} F_i^2 &= \sum_{i=0}^{n} F_i^2 + F_{n+1}^2 \\ &\underset{\text{by ind. hyp.}}{=} F_n F_{n+1} + F_{n+1}^2 \\ &= F_{n+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2}, \end{aligned}$$

which is to say the statement holds for $n + 1$. By induction the statement is true for all integers $n \geq 0$.

(c) F_{3n} is even

Base case: for $n = 0$, $F_0 = 0$ is even.

Induction hypothesis: assume true for given $n \geq 0$, i.e. F_{3n} is even.

Then

$$\begin{aligned} F_{3(n+1)} &= F_{3n+2} + F_{3n+1} \\ &= F_{3n+1} + F_{3n} + F_{3n+1} \\ &= F_{3n} + 2F_{3n+1}, \end{aligned}$$

which is even by the induction hypothesis that F_{3n} is even. By induction the statement is true for all integers $n \geq 0$.

(d) $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ (for $n \geq 1$).

Base case: for $n = 1$, $F_0F_2 - F_1^2 = 0 - 1 = (-1)^1$.

Induction hypothesis: assume true for given $n \geq 1$, i.e. $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

Then

$$\begin{aligned} F_nF_{n+2} - F_{n+1}^2 &= F_n(F_{n+1} + F_n) - F_{n+1}(F_n + F_{n-1}) \\ &= F_n^2 - F_{n+1}F_{n-1} \\ &\underset{\text{by ind. hyp.}}{=} -(-1)^n \\ &= (-1)^{n+1}, \end{aligned}$$

which is to say the statement holds for $n + 1$. By induction the statement is true for all integers $n \geq 1$.

4. You are at the bottom of a flight of stairs with n steps that you can ascend by taking one or two steps at a time. How many ways can you go up? *(If $n = 2$, you have 2 choices: take one step twice in a row, or two steps in one go. If $n = 3$, you have 3 choices: three single steps, or one single followed by one double, or one double followed by one single.)*

Let S_n denote the number of ways possible to ascend n steps in the given manner. Then S_n is the number of solutions to $x_1 + x_2 + \cdots + x_k = n$ in which $x_i \in \{1, 2\}$ and $1 \leq k \leq n$. For example, $S_3 = 3$ since $3 = 1 + 1 + 1 = 1 + 2 = 2 + 1$, while $S_4 = 5$ since $4 = 1 + 1 + 1 + 1 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 2 + 2$.

For $n \geq 1$, the ways to ascend $n+1$ steps can be divided into two types: those in which you start with 2 steps in one go, and those with just 1. If you take 2 steps to begin with, then the remaining $n - 1$ can be ascended in S_{n-1} ways, while if you take 1 step to start then there are S_n ways to complete your ascension... Hence $S_{n+1} = S_n + S_{n-1}$. This is the same recurrence as that satisfied by the Fibonacci numbers ($F_{n+1} = F_n + F_{n-1}$). The boundary conditions for this recurrence are $S_1 = 1 = F_2, S_2 = 2 = F_3$. Hence $S_n = F_{n+1}$.

In order to get to S11 for class you ascend two flights of stairs from the ground floor to the first floor, each flight consisting of 18 steps. How many times would you need to go upstairs from ground floor to first floor in order to exhaust all possible ways of going up 18 stairs?

By the previous, there are F_{19} ways to ascend a flight of 18 stairs, and you can make 2 of them each time you go upstairs from ground floor to first floor. Hence in total you would need to go upstairs at least $F_{19}/2$ times.

By direct computation (a computer is a friend here), $F_{19} = 4181$, and so 2091 ascents from ground floor to first floor would be required.