Discrete Mathematics

Exercise sheet 11

13/20 December 2016

1. A *permutation matrix* is a matrix in which each entry is either 0 or 1 and each row and column contains precisely one 1.

(a) Let $P = (p_{ij})$ be an $n \times n$ permutation matrix whose rows and columns are indexed by $[n]$. Define the function $f : [n] \to [n]$ by $f(i) = j$ precisely when $p_{ij} = 1$.

Explain briefly why f is a permutation of $[n]$.

(b) Prove that

$$P \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{f(1)} \\ x_{f(2)} \\ \cdots \\ x_{f(n)} \end{pmatrix},$$

and deduce that

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} P^T = \begin{pmatrix} x_{f(1)} & x_{f(2)} & \cdots & x_{f(n)} \end{pmatrix}.$$

(c) Prove that G and G' are isomorphic graphs if and only if a permutation matrix P exists such that

$$A_{G'} = P A_G P^T.$$

where A_G is the adjacency matrix of G and $A_{G'}$ is the adjacency matrix of G'.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 4.2, exercise 4.2.12.]

2. A *tree* is a connected graph containing no cycles as a subgraph.

(a) Prove that if $G = (V, E)$ is a graph containing no cycles and satisfying $|V| = |E| + 1$ then G is a tree.

(b) Prove that if $G = (V, E)$ is a connected graph and $|V| = |E| + 1$ then G is a tree.

(c) Prove the converse of (b).

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 5.1, Theorem 5.1.2(v) and exercise 5.1.2]