Discrete Mathematics

Exercise sheet 10

5/ 13 December 2016

1. Prove that the complement of a disconnected graph G is connected. (The complement of a graph $G = (V, E)$ is the graph $(V, \binom{V}{2} \setminus E)$.)

Let $\overline{G}$ denote the complement of G. Consider any two vertices u, v in G. If u and v are in different connected components in G then in particular there is no edge $\{u, v\}$ in G connecting them. In this case there will be an edge $\{u, v\}$ in $\overline{G}$.

If u and v are in the same connected component in G then consider any vertex w in a different connected component. (Since G is disconnected, there must be at least one other connected component than the one containing u and v.) The edges $\{u, w\}$ and $\{v, w\}$ exist in $\overline{G}$, so u and v are connected by the walk u, uw, w, wv, v. Hence any two vertices are connected by a walk joining them in $\overline{G}$, which is to say that $\overline{G}$ is connected.

What is the complement of the disjoint union of two complete graphs K_m and K_n? The complement of the complete graph K_n is the graph on n vertices having no edges (an independent set of n vertices).

The complement of the disjoint union of K_m and K_n is the complete bipartite graph $K_{m,n}$ (by definition, m independent vertices each of which is joined to every one of another set of n independent vertices).

2. Let $G = (V, E)$ be a graph.

(a) Define what is meant by a *subgraph* of G and by an *induced subgraph* of G.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap \binom{V(H)}{2}$.

A graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) = E(G) \cap \binom{V(H)}{2}$. ($H$ is a subgraph of G obtained by deleting some vertices of G and all the edges incident with these deleted vertices.)

Remark: A *spanning subgraph* of G is a subgraph H such that $V(H) = V(G)$. A spanning subgraph is obtained from G by deleting some edges of G.

(b) Show that if G contains an odd cycle as a subgraph then it also contains an odd cycle as an induced subgraph.

Suppose that C is an odd cycle contained in G as a subgraph with the further property that C is the minimum size odd cycle contained in G as a subgraph. Let $V(C) = U \subseteq V(G)$ and consider the induced subgraph H of G on vertex set U.

If $E(H) = E(C)$ then we are done. Otherwise, choose an edge $e = uv \in E(H) \setminus E(C)$ (a *chord* of the cycle C in the graph G). There are two cycles C_1 and C_2 of G sharing the edge e whose other edges partition $E(C)$ into two parts, i.e. $E(C_1) \cup E(C_2) = E(C) \cup \{e\}$ and $E(C_1) \cap E(C_2) = \{e\}$.

Exactly one of the cycles C_1 and C_2 must have odd size, because $|E(C_1)| + |E(C_2)| = |E(C)| + 2$ is odd.

But each of C_1 and C_2 has smaller size than C. We reach a contradiction to the choice of C as an odd cycle of smallest size in G. $\square$

(c) Give a counterexample to the statement that if G contains an even cycle as a subgraph then it also contains an even cycle as an induced subgraph.

For example, if G is K_4 then C_4 is a subgraph, but not induced (the only onduced subgraph on 4 vertices is K_4 itself). Also, there are no cycles of size 2 (there are no parallel edges).

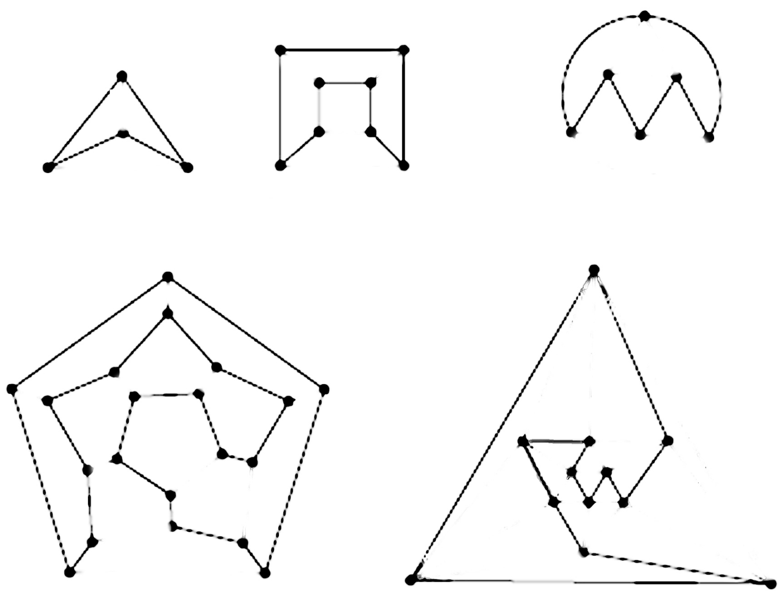
3. A *Hamiltonian cycle* in a graph G is a cycle containing all the vertices of G.
(Write down the definition of an Eulerian tour and see how this differs from the notion of a Hamiltonian cycle.)

An Eulerian tour in G is a closed walk containing each edge of G exactly once. (Vertices in general are visited multiple times: if v has degree $2d$ then it appears exactly d times in an Eulerian tour.)

A Hamiltonian cycle in G is a closed walk containing each vertex of G exactly once.

(a) Decide which of the graphs in the figure has a Hamiltonian cycle.

Here are some example Hamiltonian cycles in each graph:



(The graphs in this question are the Schlegel diagrams of the Platonic solids, each of which is Hamiltonian. The same is true of the Archimedean solids (where there may be two types of polygonal faces).

(b) Construct two connected graphs with the same score such that one has a Hamiltonian cycle while the other one does not.

If the condition of connectivity were dropped then we could simply take C_6 (Hamiltonian) and $2C_3$, i.e. two disjoint copies of C_3 (no Hamiltonian cycle), each having same score sequence $(2,2,2,2,2,2)$.

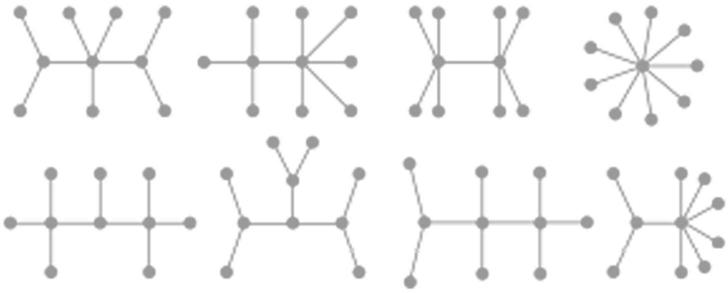
We can make both these examples connected and still preserve the property of having the same score sequence by joining a pair of non-adjacent vertices (i.e. add a chord to C_6, join together the two triangles by an edge in $2C_3$). This produces two connected graphs, each having score sequence $(3,3,2,2,2,2)$, the first with a Hamiltonian cycle, and the second without.

[In question 4 of Exercise Sheeet 8 three graphs with the same score sequence $(3,3,3,3,3,3,3,3,3,3)$ were given. The first (the Petersen graph) is not Hamiltonian, while the other two each have a Hamiltonian cycle.]

4. A *tree* is a connected graph containing no cycles as a subgraph.

A tree is *homeomorphically irreducible* if it has no vertices of degree 2. (So a path on 3 or more more vertices is not homeomorphically irreducible.)

One of the problems in the film *Good Will Hunting* (1997) is to find all homeomorphically irreducible trees with 10 vertices.



The filmmakers made a mistake: Will draws just 8 trees on the board, while in fact there exist 10 homeomorphically irreducible trees with 10 vertices. Here are the ones he draws on the board:

Find the two homeomorphically irreducible trees on 10 vertices that Will misses.