Combinatorics and Graph Theory I

Exercise sheet 9: Latin squares, Ramsey theory

3 May 2017

1.

(i) Prove that the $n \times n$ array L whose (i,j)-entry is defined by

$$L(i,j) = i + j \pmod{n}$$

is a Latin square.

(ii) Let p be a prime and $1 \leq k \leq p-1$. Prove that the $p \times p$ array L_k whose (i,j)-entry is defined by

$$L_k(i,j) = ki + j \pmod{p}$$

defines a Latin square.

(iii) Prove that when $k \neq \ell$ the Latin squares L_k and L_ℓ defined in (ii) are orthogonal.

[Adaptation of Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 9.3, exercise 5, which uses the same construction for a finite field on a prime power number of elements more generally; here the finite field is $\mathbb{Z}_p$.]

2. Use the Pigeonhole Principle to show that any finite graph has at least two vertices of the same degree.

[P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms,* Cambridge Univ. Press, 1994. Chapter 10, exercise 2]

3.

(i) Show that if $n \geq (r-1)(s-1)(t-1)+1$ then any sequence of n real numbers must contain either a strictly increasing subsequence of length r, a strictly decreasing subsequence of length s, or a constant subsequence of length t.

[*First consider the case where only* $(r-1)(s-1)$ *or fewer distinct values occur and apply the Pigenhole Principle to deduce the existence of a suitably long constant subsequence. Otherwise there are at least* $(r-1)(s-1)+1$ *distinct elements: apply the Erdős–Szekeres theorem as formulated in class.*]

(ii) Show also that the result of (i) is best possible, i.e., construct a sequence of $(r-1)(s-1)(t-1)$ real numbers with no strictly increasing subsequence of length r, no strictly decreasing subsequence of length s, and no constant subsequence of length t.

[P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms,* Cambridge Univ. Press, 1994. Chapter 10, exercise 4]

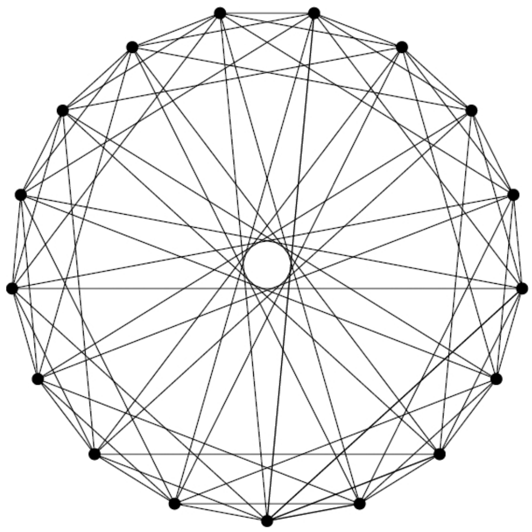

Figure 1: The graph (unique up to isomorphism) on 17 vertices with clique number 3 and independence number 3 witnessing $r(4) > 17$. If in K_{17} we colour the edges of this subgraph red and the edges of the complement of this subgraph blue then there is no monochromatic K_4. (Compare C_5, which as a subgraph of K_5 witnesses $r(3) > 5$ since K_5 with edges of a 5-cycle coloured red and edges of the complement coloured blue has no monohromatic triangle.)
(Image source: Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, Section 11.3.)

4. For $n \in \mathbb{N}$ define

$$f(n) = \min_{G:|V(G)|=n} [\alpha(G)\omega(G)],$$

where the minimum is over all graphs G with n vertices, $\omega(G)$ is the largest number of mutually adjacent vertices in G (clique number), and $\alpha(G)$ is the largest number of mutually non-adjacent vertices in G (independence number). So for example $f(2) = \min\{2 \cdot 1, 1 \cdot 2\} = 2$ (G is either a single edge K_2 or its complement).

(i) Show that for $n \in \{1, 2, 3, 4, 6\}$ we have $f(n) \geq n$.

(ii) Prove that $f(5) < 5$.

(iii) Show that $f(n)$ is nondecreasing and that it is not bounded above by a constant.

(iv)* For natural numbers n, k, $1 < k \leq n/2$ we define a graph $C_{n,k}$ as follows. We begin with C_n, i.e., a cycle of length n, and then we connect by edges all pairs of vertices that have distance at most k in C_n. Use these graphs (with a judicious choice of k) to prove that $f(n) < n$ for all $n \geq 7$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 11.2, exercises 2, 3]

5. The graph witnessing $r(4) > 17$ (see Figure 1 above) may look complicated but actually it is easy to remember. For example, it is enough to remember this: 17; 1, 2, 4, 8. Or this: quadratic residues modulo 17. Can you explain these two somewhat cryptic memory aids?

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 11.3, exercise 3]