# Combinatorics and Graph Theory I

**Exercise sheet 8: finite projective planes**

26 April 2017

1. Prove that the Fano plane is the only projective plane of order 2 (i.e. any projective plane of order 2 is isomorphic to it—define an isomorphism of set systems first).

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 9.1, exercise 1]

Choose three points not all on a line, say $x, y, z$. There are $7 \cdot 6 \cdot 4 = 168$ choices for such a triple $(x, y, z)$ (Choose any pair of points $(x, y)$, and then there are 4 points not on the line $\overline{xy}$ to choose from – this is the only freedom we have in constructing a projective plane on a given set of seven points, as we shall see.)

The three lines $\overline{yz}, \overline{xz}, \overline{xy}$ each determine a third point, and these three points are distinct, since lines cannot meet in two or more points. Call these points $a, b, c$ respectively (they are determined by our choice of $x, y, z$).

The line $\overline{zc}$ contains a third point not among $x, y, a, b$ since two lines cannot share more than one point (for example, if $x$ were on $\overline{zc}$ then there would be an overlap of two points with line $\overline{xz}$). Call this new point $d$ (the last of the seven points).

The point $x$ so far has two lines though it, $\{x, y, c\}$ and $\{x, b, z\}$; the points $x$ and $a$ determine the third line as $\{x, a, d\}$ (no other choice for the third point $d$ is possible since other choices always lead to a pair of lines meeting in two points). Likewise, the point $y$ so far has two lines though it, $\{y, z, a\}$ and $\{x, y, c\}$; the points $y$ and $b$ determine the third line as $\{y, b, d\}$ (any other choice leading to a pair of lines sharing two points).

The points $x, y, z, d$ now each have three lines going through them. The points $a, b, c$ each have two lines going through them. The only choice for the line determined by $a, b$ that respects the axiom that any two lines meet in just one point is to take the line $\{a, b, c\}$.

The projective plane constructed is isomorphic to the Fano plane on point set $\{1, 2, 3, 4, 5, 6, 7\}$ and lines $\{1, 6, 7\}$, $\{2, 5, 7\}$, $\{3, 4, 7\}$, $\{1, 2, 4\}$, $\{2, 3, 6\}$, $\{1, 3, 5\}$, $\{4, 5, 6\}$ by the bijection between points defined by the following:

| $x$ | $y$ | $z$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 6 | 5 | 4 | 7 |

There are 168 choices for the first three entries under $x, y, z$, the remaining points being then determined. We have chosen $1, 2, 3$ here.

An isomorphism of projective planes is a permutation equivalence of their incidence matrices; equivalently, an isomorphism between their incidence graphs that permutes the points among themselves and lines among themselves. (A permutation involving a swap of points and lines includes a duality; the Fano plane is self-dual.) The incidence matrix for the plane constructed on $\{x, y, z, a, b, c, d\}$ is permutation equivalent to the incidence matrix for the Fano plane.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

by using the permutation described in the table above

2. Let $(X, \mathcal{L})$ be a finite projective plane with set of points $X$ and set of lines $\mathcal{L}$. Let $r$ be the *order* of $(X, \mathcal{L})$, defined as the number of points less one in any given line, i.e., $r = |L| - 1$ for $L \in \mathcal{L}$. The *incidence graph* (or *Levi graph*) of $(X, \mathcal{L})$ is the bipartite graph on $X \cup \mathcal{L}$ with an edge joining $x$ to $L$ precisely when $x \in L$.

(i) The *girth* of a graph with at least one cycle is the smallest positive integer $g$ for which there is a $g$-cycle. (Thus for instance a triangle-free graph has girth at least 4.) A *$k$-regular graph* is a graph in which each vertex has degree $k$.

Show that a $k$-regular graph with girth $2m + 1$ must have at least $1 + k + k(k-1) + \cdots + k(k-1)^{m-1}$ vertices, and that a $k$-regular graph with girth $2m$ must have at least $2[1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{m-1}]$ vertices.

Consider first a $k$-regular graph $G$ of odd girth $2m + 1$. Choose an arbitrary vertex $v$. Consider the subgraph of $G$ consisting of vertices connected to $v$ by a path of length at most $m$.

This is a tree with $v$ as its centre, and in which each non-leaf vertex has degree $k$, and in which each of the leaves is distance $m$ from the root: It is connected since any pair of vertices can be joined by a path passing through $v$. It is acyclic because the only way a cycle can be formed is if two paths from $v$ meet in another vertex, but this would create a cycle of length less than $2m + 1$, contradicting the girth condition.

This tree has $1 + k + k(k-1) + ... + k(k-1)^{m-1}$ vertices (for $0 < i \leq m$, the number of vertices at distance $i$ from the root is $k(k-1)^{i-1}$, as can be proved by induction).

For a $k$-regular graph of even girth $2m$, choose a pair of adjacent vertices $u$ and $v$ and consider the two trees centred at $u$ and at $v$, in which the centre $u$ of one tree has degree $k - 1$, the centre $v$ of the other tree also has degree $k - 1$, each other non-leaf vertex has degree $k$, and the leaves in either tree are distance $m - 1$ from the root vertex. The tree containing $u$ as centre does not contain $v$, and conversely the tree containing $v$ as centre does not contain $u$.

Each of the two trees has $1 + (k-1) + (k-1)^2 + ... + (k-1)^{m-1}$ vertices, since there are $(k-1)^i$ vertices at distance $i$ from the root. Each are trees in having no cycles because a cycle in either would have length at most $2m - 2$, and furthermore, the two trees do not overlap in any vertices for otherwise a cycle of length at most $2m - 1$ would be formed (involving the edge $uv$).

The two vertex-disjoint trees centred at $u$ and at $v$ are subgraphs of $G$ and together account for $2[1 + (k-1) + (k-1)^2 + ... + (k-1)^{m-1}]$ vertices.

(ii) Show that the incidence graph of $(X, \mathcal{L})$ is an $(r{+}1)$-regular graph of girth 6 which attains the lower bound given in (i) for $m = 3$. *(Thus the incidence graph of a projective plane of order $r$ has the minimum number of vertices among all $(r{+}1)$-regular graphs of girth 6.)*

The incidence graph of a projective plane of order $r$ has $2[1+r+r^2]$ vertices, is bipartite with one half of the vertices representing points the other half lines and $(r+1)$-regular since each point lies on $r+1$ lines and each lines contains $r+1$ points.The girth is even (since every cycle is even in a bipartite graph), and greater than 4 since a 4-cycle would correspond to a pair of lines meeting in two common points, and equal to 6 since three points not all on a line give a triangle in the plane, corresponding to a 6-cycle in the incidence graph.

Taking $k = r+1$ and $m = 3$ in (i), a $(r+1)$-regular graph of girth 6 has at least $2[1+r+r^2]$ vertices, and this is achieved by the incidence graph of a projective plane of order $r$.

[N. Biggs, *Discrete Mathematics,* rev. ed., 1989. 8.8, exercises 18 (and 20), and 16.10, exercise 10.]

3. Let $X$ be a finite set and $\mathcal{L}$ a system of lines (subsets of $X$) satisfying conditions (P1) and (P2), and the following condition:

(P0') There exist at least two distinct lines having at least three points each.

Prove that any such $(X, \mathcal{L})$ is a finite projective plane. [*Hint*: By (P0') and (P1) the symmetric difference of the two such lines contains at least four points. Show these give a set $F$ satisfying (P0).]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 9.1, exercise 4]

Let $L_1, L_2$ be lines satisfying (P0'), i.e., $|L_1| \geq 3, |L_2| \geq 3$. By (P1) $|L_1 \cap L_2| = 1$, so that $|L_1 \Delta L_2| = |L_1 \cup L_2| - |L_1 \cap L_2| = |L_1| + L_2| - 2|L_1 \cap L_2| \geq 3+3-2 = 4$. Let $F = \{a_1, b_1, a_2, b_2\} \subseteq L_1 \Delta L_2$, with $\{a_1, b_1\} \subseteq L_1 \setminus L_2$ and $\{a_2, b_2\} \subseteq L_2 \setminus L_1$.

We need to show that $|F \cap L| \leq 2$ for all lines $L$. This is true for $L = L_1$ ($F \cap L_1 = \{a_1, b_1\}$) and $L = L_2$ ($F \cap L_2 = \{a_2, b_2\}$). If a line meets $F$ in three or more points then it either passes through $a_1$ and $b_1$ (in which case by (P2) it is determined as the line $L_1$) or it passes through $a_2$ and $b_2$ (in which case by (P2) it is determined as the line $L_2$). Hence any other line must meet $F$ in at most two points. Therefore $F$ is a quadrangle, satisfying (P0).

4.

(i) Find an example of a set system $(X, \mathcal{L})$ on a non-empty finite set $X$ that satisfies conditions (P1) and (P2) but does not satisfy (P0).

For example, take the only line to be $X$ (containing all points), i.e. $\mathcal{L} = \{X\}$.

(ii) Describe *all* set systems $(X, \mathcal{L})$ on non-empty finite set $X$ satisfying conditions (P1) and (P2) but not (P0).

[*Hint*: By question 3, it may be assumed that there is a most one line containing three or more points.]

If (P1) and (P2) hold and there are two lines with three or more points then by the previous question (P0) is also satisfied.

Let $L$ be the single line that contains three or more points. Other lines must have one or two points at most.

If there is a line $\{a\}$ consisting of just one point $a$, then by (P1) every other one-point line must contain $a$, so there is at most one such line. Supposing $\{a\}$ is a single-point line, if there is a line with two points, then by (P1) one of these points is $a$. Further, a two-point line $\{a, b\}$ meets the line $L$ in a unique point, which by (P1) must be $b$.

So we have the following possibilities for a projective geometry $(X, \mathcal{L})$ on a finite set $X$ of three or more elements:

- $\mathcal{L} = \{X\}$ (no 1- or 2-point lines)
- $\mathcal{L} = \{\{a\}, X\}$ for some $a \in X$ (no 2-point line)
- $\mathcal{L} = \{\{a\}, X \setminus \{a\}\} \cup \{\{a, b\} : b \in X \setminus \{a\}\}$ for some $a \in X$ (no 1-point line)

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 9.1, exercise 3]