Combinatorics and Graph Theory I

Exercise sheet 7: Spanning trees and double counting

19 April 2017

1. Let $\tau(G)$ denote the number of spanning trees of a connected graph G. Cayley's formula states that $\tau(K_n) = n^{n-2}$ for $n \geq 2$. Let K_n^- denote a graph isomorphic to K_n with one edge removed.

Find a formula for $\tau(K_n^-)$.

[*Hint*: the number of spanning trees containing a given edge of K_n is by symmetry the same for all edges.]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 8.1, exercise 2]

We will count pairs (T, e) such that T is a spanning tree of K_n that contains a fixed edge $e \in E(K_n)$. *(You can think of an incidence matrix with rows indexed by spanning trees of K_n and columns by edges of K_n. We shall sum the total number of 1s in this incidence matrix by first summing columns, and then totting up these column sums, and then after this go by row sums first instead, which are then totted up. Either way we get the same total.)*

As remarked in the hint, the number of T containing a given edge e is independent of e, and hence just a function of n; say there are C_n such spanning trees.

Then, letting e range over $E(K_n)$ *(the columns of the (T, e)-incidence matrix)* there are $\binom{n}{2}C_n$ pairs (T, e) in which T is a spanning tree of K_n and $e \in T$.

On the other hand, the number of pairs (T, e) such that T is a spanning tree of K_n and $e \in T$ is also equal to $(n-1)\tau(K_n)$, since there are $(n-1)$ edges $e \in T$ *(the rows of the (T, e)-incidence matrix)*.

Thus $\binom{n}{2}C_n = (n-1)C_n = (n-1)n^{n-2}$, using Cayley's formula $\tau(K_n) = n^{n-2}$ in the second equality, and we have $C_n = 2n^{n-3}$.

$\tau(K_n^-)$ is equal to $\tau(K_n) - C_n$, since a spanning tree of K_n not containing e is a spanning tree of $K_n - e \cong K_n^-$. From this,

$$\tau(K_n^-) = n^{n-2} - 2n^{n-3} = n^{n-3}(n-2).$$

2.

(i) Determine natural numbers a and b with $a + b = n$ for which the product ab is maximized.

By the AGM inequality $\sqrt{ab} \leq \frac{a+b}{2} = n/2$, i.e. $ab \leq n^2/4$, with equality if and only if $a = b$. Therefore take $a = b = \frac{n}{2}$ when n is even, and $a = \frac{n\pm1}{2}, b = \frac{n\mp1}{2}$ when n is odd.

Alternative argument: We may suppose $0 < a \leq b \leq n$ and $a + b = n$. Let $d = \frac{b-a}{2} \geq 0$. Then $ab = (\frac{n}{2} - d)(\frac{n}{2} + d) = \frac{n^2}{4} - d^2$ so to maximize the product ab given $a + b = n$, $a \leq b$, is the same as minimizing the difference $b - a$, i.e. make this difference as close to zero as possible. Thus take $a = b = \frac{n}{2}$ when n is even, and $a = \frac{n-1}{2}, b = \frac{n+1}{2}$ when n is odd.

(ii) For natural numbers k and n, determine all values of natural numbers $a_1, \ldots, a_k$ satisfying $\sum_{i=1}^{k} a_i = n$ such that the product $a_1 a_2 \cdots a_k$ is maximized.

By the AGM inequality[1]

$$(a_1 a_2 \cdots a_k)^{\frac{1}{k}} \leq \frac{a_1 + a_2 + \cdots + a_k}{k},$$

with equality if and only if $a_1 = a_2 = \cdots = a_k$. Therefore, for integers $a_1, \ldots, a_k$ with sum n we take $a_i = \lceil \frac{n}{k} \rceil$ for $i = 1, \ldots r$, where $n = km + r$ for $0 \leq r < k$ and $a_i = \lfloor \frac{n}{k} \rfloor$ for $i = r+1, \ldots, n$. (When n is a multiple of k, $r = 0$ and we take $a_i = n/k$ for each $i = 1, \ldots, k$.).

(iii) A complete k-partite graph $K(V_1, V_2, \ldots, V_k)$ on a vertex set V is determined by a partition $V_1, \ldots, V_k$ of the set V, in which edges are pairs $\{x, y\}$ of vertices such that x and y lie in different classes of the partition. Formally, $K(V_1, \ldots, V_k) = (V, E)$, where $\{x, y\} \in E$ exactly if $x \neq y$ and $|\{x, y\} \cap V_i| \leq 1$ for all $i = 1, \ldots, k$. Using part (ii), prove that the maximum number of edges of a complete k-partite graph on a given vertex set corresponds to a partition with almost equal parts, i.e. one with $||V_i| - |V_j|| \leq 1$ for all i, j. How many edges are there in such a graph $K(V_1, \ldots, V_k)$?

Let $n = |V|$ and $a_i = |V_i|$. Then $a_1 + \cdots + a_k = n$ and the number of edges in $K(V_1, V_2, \ldots, V_k)$ is

$$\begin{aligned}\sum_{1 \leq i < j \leq k} a_i a_j &= \frac{1}{2}\left((\sum_{1 \leq i \leq k} a_i)^2 - \sum_{1 \leq i \leq k} a_i^2 \right) \\ &= \frac{1}{2} n^2 - \frac{1}{2} \sum_{1 \leq i \leq k} a_i^2.\end{aligned}$$

So we wish to minimize the sum $\sum_{1 \leq i \leq k} a_i^2$ subject to $\sum_{1 \leq i \leq k} a_i = n$ and $a_i \in \mathbb{N}$.

By the Cauchy-Schwarz inequality,

$$\left(\sum_{1 \leq i \leq k} a_i \cdot 1 \right)^2 \leq \sum_{1 \leq i \leq k} a_i^2 \sum_{1 \leq i \leq k} 1^2$$

i.e.

$$n^2/k \leq \sum_{1 \leq i \leq k} a_i^2$$

with equality if and only if $a_1 = a_2 = \cdots = a_k$. Therefore we should take (as in (ii)[2]) $a_i = \lceil \frac{n}{k} \rceil$ for $i = 1, \ldots r$, where $n = km + r$ for $0 \leq r < k$ and $a_i = \lfloor \frac{n}{k} \rfloor$ for $i = r+1, \ldots, n$. Then $|a_i - a_j| \leq 1$ for each $1 \leq i, j \leq k$, i.e. $||V_i| - |V_j|| \leq 1$.

Alternative argument: Set $a_i = |V_i$ as before. Assume for the sake of contradiction that there are i, j such that $a_j - a_i \geq 2$ in a k-partite graph $G = K(V_1, \ldots, V_k)$ with maximum number of edges m. Consider now the graph created from G by taking a vertex $v \in V_j$ and adding it V_i. This switch changes the number of edges by

$$(a_i + 1)(a_j - 1) - a_i a_j = a_j - a_i - 1 \geq 1$$

contradicting minimality of m. Hence the sizes of two sets cannot differ in size by more than 1.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed., 4.7, exercises 1,2 and 3]

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[1]For a proof of the AGM inequality for general k, see for example `https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means`

[2]In Matoušek & Nešetřil they state that part (ii) should be used to solve this part (iii), but currently I don't see how exactly, and so have used an alternative route.

3. Prove that for any $t \geq 2$, the maximum number of edges of a graph on n vertices containing no $K_{2,t}$ as a subgraph is at most

$$\frac{1}{2}\left(\sqrt{t-1}n^{3/2}+n\right).$$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, 2nd ed. 7.3, exercise 1]

We count the number of pairs $(v, \{u, u'\})$ such that $v \in V$, $\{u, u'\} \in \binom{V}{2}$ and $uv, u'v \in E(G)$ in two ways.

First, for fixed v there are $\binom{\deg(v)}{2}$ choices for $\{u, u'\}$ such that $uv, u'v \in E(G)$.

Second, for fixed $\{u, u'\}$ there are at most $(t-1)$ distinct choices for v such that $uv, u'v \in E(G)$. This is because t such vertices v would give a copy of $K_{2,t}$, which is excluded.

Since there are $\binom{n}{2}$ choices for $\{u, u'\} \in \binom{V}{2}$, we therefore have

$$\sum_{v\in V}\binom{\deg(v)}{2} \leq (t-1)\binom{n}{2}.$$

Using $\binom{d}{2} \geq \frac{1}{2}(d-1)^2$ and $\binom{n}{2} < \frac{1}{2}n^2$, this gives

$$\sum_{v\in V}(\deg(v)-1)^2 < (t-1)n^2,$$

and then

$$\sum_{v\in V}(\deg(v)-1)^2 < (t-1)n^2,$$

On the other hand, by the (square of the) Cauchy-Schwarz inequality,

$$\left(\sum_{v\in V}(\deg(v)-1)\cdot 1\right)^2 \leq \sum_{v\in V}(\deg(v)-1)^2 \sum_{v\in V} 1^2,$$

from which

$$2|E| - n < \sqrt{t-1}n \cdot \sqrt{n},$$

i.e.

$$|E| < \frac{1}{2}(\sqrt{t-1}n^{3/2}+n).$$

Note: The simplification of $\binom{d}{2}$ to $\frac{1}{2}(d-1)^2$ in the course of the proof means that the inequality bounding $|E|$ is not sharp: we replace $\sum_{v\in V}\binom{\deg(v)}{2}$ by $\sum_{v\in V}\frac{1}{2}(\deg(v)-1)^2$, which is less by a difference of $\frac{1}{2}\sum_{v\in V}(\deg(v)-1) = |E| - \frac{1}{2}n$. Therefore we cannot expect to achieve the upper bound. However, as shown in Exercise 9.4.2 in Matoušek & Nešetřil, for $t = 2$ (excluding $K_{2,2}$ as a subgraph) the dominant term $\frac{1}{2}n^{3/2}$ is attainable when n is a prime power.