Combinatorics and Graph Theory I

Exercise sheet 6: Graph connectivity

12 April 2017

1. Let $\delta(G)$ denote the minimum degree of graph G.

(i) Define the parameters $\kappa(G)$ and $\lambda(G)$.

A graph G is 1-connected if it is connected.

For $k \geq 2$, a graph $G = (V, E)$ is k-connected if $|V| > k$ and there is no $U \subset V$ of size $|U| < k$ such that $G - U$ is disconnected.

The *connectivity* of G is defined as

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$

It is also equal to the minimum k such that there is $U \subset V$ of size $|U| = k$ such that $G - U$ is disconnected, with the exception of G on k vertices, $G \cong K_{k+1}$, $\kappa(K_{k+1}) = k$, for which removing k vertices leaves a single vertex, which is trivially connected.

A graph G is 1-edge-connected if it is connected. For $k \geq 2$, a graph $G = (V, E)$ is k-edge-connected if $|V| > 1$, and there is no $F \subset E$ of size $|F| < k$ such that $G - F$ is disconnected. The *edge-connectivity* of G is defined as

$$\lambda(G) = \max\{k : G \text{ is } k\text{-edge-connected}\}.$$

It is also equal to the minimum k such that there is $F \subset E$ of size $|F| = k$ such that $G - F$ is disconnected.

(ii) Prove that

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

for a graph G on more than one vertex.

[Bollobás, *Modern Graph Theory*, III.2.]

First we prove that $\lambda(G) \leq \delta(G)$. Take a vertex v of minimum degree $\delta(G)$ and set F to be the set of edges incident with v. Then $G - F$ is disconnected (since v is isolated) and $|F| = \delta(G)$. This implies G is at most $\delta(G)$-edge-connected. Hence $\lambda(G) \leq \delta(G)$.

Second we prove that $\kappa(G) \leq \lambda(G)$. If $\lambda(G) = 1$ then G is connected and $\kappa(G) = 1 = \lambda(G)$. Suppose then $\lambda(G) = k \geq 2$. For G the complete graph on $k+1$ vertices we have $\kappa(G) = k = \lambda(G)$. So we may assume G has at least $k+2$ vertices. Since $\lambda(G) = k$, there is a set of edges $F = \{u_1v_1, \ldots, u_kv_k\}$ disconnecting G, in which we may assume notation has been chosen so that $u_1, \ldots, u_k$ belong to the same component C of $G - F$ (minimality of k for edge-cut size means that $G - F$ has two components only: one containing the u_i, the other the v_i). If $G - \{u_1, \ldots, u_k\}$ is disconnected then $\kappa(G) \leq k$. Suppose then that $G - \{u_1, \ldots, u_k\}$ is connected. Then $u_1, \ldots, u_k$ form the whole vertex set of C. It follows that each vertex u_i has at most k neighbours, namely some of the other vertices

$u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_k$, and the vertex v_i. The degree of u_i must in fact equal k since $\delta(G) \geq \lambda(G) = k$. Deleting the neighbours of u_i disconnects the graph (here we use that there are at least $k+2$ vertices so that the graph is indeed disconnected by isolating the vertex u_i). Hence $\kappa(G) \leq k = \lambda(G)$ here too.

(iii) Let k and ℓ be integers with $1 \leq k \leq \ell$.

(a) Construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

In fact we construct a graph with $\delta(G) = d \geq \ell$ and $|V(G)| = n > 2d$.

Let $U = \{u_1, \dots, u_{d+1}\}$ and $V = \{v_1, \dots, v_{n-d-1}\}$ be disjoint sets of vertices. Let G be the graph on vertex set $U \cup V$ such that $G[U] \cong K_{d+1}$ and $G[V] \cong K_{n-d-1}$, and with further edges $u_1v_1, \dots, u_kv_k$ plus $\ell - k$ further edges u_iv for $v \in V$. Then G has n vertices, minimum degree d, connectivity k (remove $\{u_1, \dots u_k\}$) and edge-connectivity ℓ (remove the edges between U and V).

(b) Construct a graph G with $\kappa(G) = k$ and $\kappa(G - v) = \ell$ for some vertex v.

Let $U = \{u_1, \dots, u_{\ell+1}\}$ and $v \notin U$.

Let G be the graph on vertex set $U \cup \{v\}$ such that $G[U] \cong K_{\ell+1}$ and $u_1v, u_2v, \dots, u_kv$ are the only other edges. Then the vertex cut $\{u_1, u_2, \dots u_k\}$, producing an isolated vertex v, shows that $\kappa(G) = k$ (as there are no smaller vertex cuts) while $G - v \cong K_{\ell+1}$ has connectivity ℓ.

(a) Construct a graph G with $\lambda(G - u) = k$ and $\lambda(G - uv) = \ell$ for some edge uv.

Let $U = \{u_1, \dots, u_\ell\} \cup \{u\}$ and $V = \{v_1, \dots, v_\ell\} \cup \{v\}$ be disjoint sets of vertices.

Let G be the graph on vertex set $U \cup V$ such that $G[U] \cong K_{\ell+1} \cong G[V]$, uv is an edge, and $u_1v, u_2v, \dots, u_kv$ and $uv_1, \dots, uv_{\ell-k}$ are the remaining edges.

Then $G - uv$ has edge cut $\{u_1v, u_2v, \dots, u_kv\} \cup \{uv_1, \dots, uv_{\ell-k}\}$ of size ℓ and no smaller edge cuts, so $\lambda(G - uv) = \ell$. The graph $G - u$ has edge cut $\{u_1v, u_2v, \dots, u_kv\}$ of size k and no smaller ones, so $\lambda(G - u) = k$.

[Bollobás, *Modern Graph Theory*, III.6, exercise 11]

2. Given $U \subset V(G)$ and a vertex $x \in V(G) - U$, an $x - U$ *fan* is a set of $|U|$ paths from x to U any two of which have exactly the vertex x in common. Prove that a graph G is k-connected iff $|G| \geq k + 1$ and for any $U \subset V(G)$ of size $|U| = k$ and vertex x not in U there is an $x - U$ fan in G.

[*Given a pair (x, U), add a vertex u to G and join it to each vertex in U. Check that the new graph is k-connected if G is. Apply Menger's theorem for x and u.*]

[Bollobás, *Modern Graph Theory*, III.6 exercise 13]