Combinatorics and Graph Theory I

Exercise sheet 5: Bipartite matching

5 April 2017

1. An *augmenting path* with respect to a matching M in a bipartite graph G is a path P in G which starts at an unmatched vertex and then contains alternately an edge not in M and an edge in M and ends in another unmatched vertex. The symmetric difference of the edges of P and the edges of M is again a matching and covers two more vertices than M (the endpoints of P).

(i) Let M be a matching in a bipartite graph G. Show that if M contains fewer edges than some other matching N in G then G contains an augmenting path with respect to M.

[Consider the symmetric difference of M and N.]

The spanning subgraph $M \triangle N$ has vertices of degree $0, 1$ or 2. Its connected components are thus cycles or paths. In a cycle the edges alternate between M and N, and so contain an equal number of edges in M as in N. Likewise, on a path edges alternate beween M and N. Since N has more edges in total than M, there must then be some path in which there are more edges in N than in M. Necessarily in this case the endpoints of the path are incident with edges of N, which are therefore not incident with edges of M. This path is thus an augmenting path for M.

(ii) Describe an algorithm based on (i) that finds a matching of maximum cardinality in any given biparite graph. [*Fine details not required; you may assume there is an oracle that provides you with an augmenting path when one exists.*]

1. Begin with any matching M (one edge alone will serve)
2. Search for an augmenting path for M
3. If an augmenting path is found, construct a larger matching M' by switching edges along this path, and return to step 2. with M' in place of M
4. If no augmenting path can be found, stop: M is a maximum matching.

See N. Biggs, *Discrete Mathematics*, 10.5 for a description of how an augmenting path can be found by breadth-first-search, which starts at an unmatched vertex and builds a tree of 'partial' alternating paths starting from it, which are turn by turn extended until either another unmatched vertex is found or all vertices are visited.

2.

(a) Use Hall's condition to show that the bipartite graph in Figure 1 has no complete matching.

$N(\{x_1, x_3, x_4\}) = \{y_2, y_4\}$ violates Hall's condition for a complete matching from the xs to the ys. (Also $N(\{x_1, x_2, x_3, x_4\}) = \{y_2, y_4, y_5\}$.)

$N(\{y_1, y_3\}) = \{x_5\}$ violates Hall's condition for a complete matching from the ys to the xs.

(b) Let M be the matching $\{x_3y_2, x_4y_4, x_5y_5\}$ denoted by heavier lines in Figure 1.

(i) Find an augmenting path for M beginning at x_2. $x_2y_5x_5y_1$ is an example.

(ii) Use it to construct a matching M' with four edges. $M' = \{x_2y_5, x_3y_2, x_4y_4, x_5y_1\}$

(iii) Check that there is no augmenting path for M'.

(iv) Is M' a maximum matching? (i.e. are there any matchings with more than four edges?) Yes: any larger would be a complete matching, which we know does not exist by (a).

[Biggs, *Discrete Mathematics*, exercises 10.4.1 and 10.4.2]

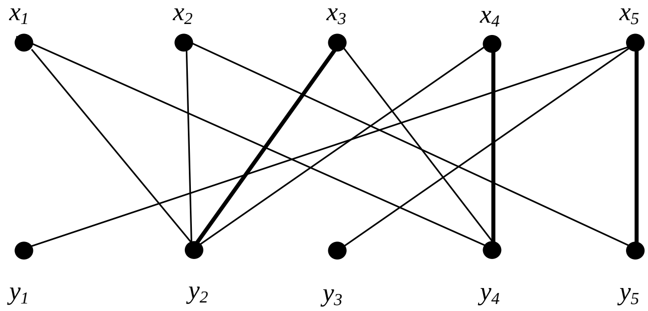

Figure 1: Bipartite graph for Exercise 2.

3. Let G be a bipartite graph with vertex sets V_1 and V_2. Let U be the set of vertices of maximal degree (i.e., the degree of each vertex in U is the maximum degree of G). Show that there is a complete matching from $U \cap V_1$ into V_2.

[Bollobás, *Modern Graph Theory*, III.6 exercise 21.]

Let k be the maximum degree of G and let U be the set of vertices in G of degree k.

Let $S \subseteq U \cap V_1$. There are $k|S|$ edges with an endpoint in S. Each vertex in V_2 has degree $\leq k$. So there are at most $k|N(S)|$ edges with an endpoint in S. It follows that $k|S| \leq k|N(S)|$, so that Hall's condition $|S| \leq |N(S)|$ is satisfied for each $S \subseteq U \cap V_1$, so there is a complete matching of the bipartite graph with bipartition $(U \cap V_1, V_2)$.

When $U = V_1 \cup V_2$ we obtain the following theorem, whose proof is given to exhibit the parallel:

Every regular bipartite graph has a perfect matching.

Proof: Let G be a k-regular bipartite graph with bipartition (V_1, V_2). Let $S \subseteq V_1$ and let t be the number of edges with one end in X. Since every vertex in S has degree k, it follows that $k|S| = t$. Similarly, every vertex in $N(S)$ has degree k, so t is less than or equal to $k|N(S)|$. It follows that $|S|$ is at most $|N(S)|$. Thus, by Hall's Theorem, there is a matching covering V_1, or equivalently, every maximum matching covers V_1. By a similar argument, we find that every maximum matching covers V_2, and this completes the proof.