Combinatorics and Graph Theory I

Exercise sheet 4: Flows in directed graphs

15 March 2017

2. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a digraph with source s and sink t and suppose $\phi : \overrightarrow{E} \to \mathbb{R}$ is a function, not necessarily a flow.

(a) Show that

$$\sum_{x \in V} \phi(x, V \setminus x) = \sum_{x \in V} \phi(V \setminus x, x).$$

It will help to relabel the dummy variable $x \in V$ in the right-hand sum as $y \in V$. We are asked to show that

$$\sum_{x \in V} \phi(x, V \setminus x) = \sum_{y \in V} \phi(V \setminus y, y).$$

Any given arc $(x, y) \in \overrightarrow{E}$ with $y \in V \setminus x$ contributes once to the sum on the left-hand side, giving a term $\phi(x, y)$; and the same arc contributes once to the right-hand side as $x \in V \setminus y$, again the term it contributes being $\phi(x, y)$.

Hence the two sums are equal, as they each range over the set of all (non-loop) arcs in $\overrightarrow{G}$:

$$\bigcup\{(x, y) : x \in V, y \in V \setminus x\} = \bigcup\{(x, y) : y \in V, x \in V \setminus y\}$$

(b) Deduce from (a) that if ϕ is a flow then the net flow out of s is equal to the net flow into t:

$$\phi(s, V \setminus s) - \phi(V \setminus s, s) = \phi(V \setminus t, t) - \phi(t, V \setminus t).$$

By definition, if ϕ is a flow then $\phi(x, V \setminus x) = \phi(V \setminus x, x)$ for all $x \in V \setminus \{s, t\}$. Hence by (a), after cancelling all but the terms indexed by s and t,

$$\phi(s, V \setminus s) + \phi(t, V \setminus t) = \phi(V \setminus s, s) + \phi(V \setminus t, t).$$

Rearranging this equation give the result.

3. Suppose $S \subseteq \overrightarrow{E}$ is a set of edges after whose deletion there is no flow from s to t with strictly positive value. Prove that S contains a cut separating s from t, i.e., there is $X \subset V$ with $s \in X$ and $t \notin X$ such that $\overrightarrow{E}(X, V \setminus X) \subseteq S$.

[Bollobás, *Modern Graph Theory*, II.6 exercise 1]

Set the capacity of arcs in S equal to 0. By the max-flow min-cut theorem there is a flow with value at least equal to the minimum size of a cut separating s and t in $\overrightarrow{G} \setminus S$. This minimum size is zero by the assumption that there is no flow from s to t with strictly positive value. In other words, s and t are already separated in $\overrightarrow{G} \setminus S$. Define X to be all the vertices reachable by a directed path from s. Then $s \in X$ and by what we have just proved $t \notin X$ (otherwise we could send a positive flow along this path from s to t). From this it follows that S must contain $\overrightarrow{E}(X, V \setminus X)$.

4. Let $f : \overrightarrow{E} \to \mathbb{R}^+$ be a flow on a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ with source s, sink t and capacity function $c : \overrightarrow{E} \to \mathbb{R}^+$.

(a) Define the *value* of the flow f.

The value of f is defined as the common value of $f(s, V \setminus s) - f(V \setminus s, s)$ and $f(V \setminus t, t) - f(t, V \setminus t)$, i.e., the net flow going out of the source s, which is the same as the net flow going into the sink t.

(b) Suppose that $X \subseteq V$ contains s but not t. Show that the value of f is also equal to

$$f(X, V \setminus X) - f(V \setminus X, X).$$

We have

$$\begin{aligned} f(s, V \setminus s) - f(V \setminus s, s) &= f(s, V) - f(V, s) \\ &= \sum_{x \in X} [f(x, V) - f(V, x)] \\ &= \sum_{x \in X} [f(x, V \setminus X) - f(V \setminus X, x)] \\ &= f(X, V \setminus X) - f(V \setminus X, X) \end{aligned}$$

where to get to the second line we use the flow condition $f(x, V) - f(V, x) = 0$ for each $x \in V \setminus \{s, t\}$ and to get to the third line we use the fact that to each arc $(x_1, x_2) \in X \times X$ there is a contribution of $f(x_1, x_2) - f(x_1, x_2) = 0$ to $\sum_{x \in X}[f(x, V) - f(V, x)]$. An alternative way to write the same sequence of equalities is

$$\begin{aligned} f(s, V \setminus s) - f(V \setminus s, s) &= f(s, V) - f(V, s) \\ &= \sum_{x \in X} [f(x, V) - f(V, x)] \\ &= f(X, V) - f(V, X) \\ &= f(X, X) + f(X, V \setminus X) - [f(X, X) + f(V \setminus X, X)] \\ &= f(X, V \setminus X) - f(V \setminus X, X) \end{aligned}$$

(c) Using (b) and the fact that $0 \leq f(x, y) \leq c(x, y)$ for each $(x, y) \in \overrightarrow{E}$ prove that the value of f is at most equal to the capacity of a cut $\overrightarrow{E}(X, V \setminus X)$ separating s from t.

Since $0 \leq f(x, y) \leq c(x, y)$ for each arc (x, y), we then have $0 \leq f(V \setminus X, X)$ and $f(X, V \setminus X) \leq c(X, V \setminus X)$, so that

$$f(X, V \setminus X) - f(V \setminus X, X) \leq c(X, V \setminus X).$$

Remark: Equality holds if and only if $f(x, y) = c(x, y)$ for each arc (x, y) with $x \in X, y \in V \setminus X$ and $f(y, x) = 0$ for each arc (y, x) with $x \in X, y \in V \setminus X$: f reaches capacity going from X to $V \setminus X$ and does not use arcs going back from $V \setminus X$ to X.

[Bollobás, *Modern Graph Theory*, III.6 exercise 2.]

5. Let f be a flow on a network comprising digraph $\overrightarrow{G}$, source s, sink t, and capacity function $c : \overrightarrow{E} \to \mathbb{R}^+$.

A *circular flow* in f is a directed cycle in $\overrightarrow{G}$ such that $f(x, y) > 0$ for each arc (x, y) in the cycle.

By successively reducing the number of circular flows in a given flow f of maximum value prove that there is a maximal flow f^* without circular flows in which $f^*(V \setminus s, s) = 0$ and $f(t, V \setminus t) = 0$.

[Bollobás, *Modern Graph Theory*, III.6 exercise 4.]

If f takes minimum value $a > 0$ on a directed cycle then by subtracting a from each edge of the cycle we obtain another flow with the same value but equal to zero for at least one edge on the cycle. Recursively reducing positive cycles in this way we obtain a flow f^* with no positive cycles (an *acyclic flow*).

Given an acyclic flow f^* of value v, there must be a directed path from s to t along which all flow is positive, because there are no positive cycles. (For every vertex not s or t with a positive incoming arc there is a positive outgoing arc, by the Kirchhoff rule and by nonnegativity of f^*. Starting from s, following arcs with positive flow either you reach t or you traverse a directed cycle, and the latter cannot happen.) Decrementing the flow on each edge of this path yields another acyclic flow of value $v - 1$. Recursively we decompose f^* as a sum of flows of value 1 along directed paths from s to t. It follows that f^* is zero on any arcs going into s and any arcs going out of t.