Combinatorics and Graph Theory I

Exercise sheet 4: Flows in directed graphs

15 March 2017

1. Sketch the network whose vertices are s, a, b, c, d, t and whose arcs and capacities are

$(x,y):$	(s,a)	(s,b)	(a,b)	$(a.c)$	(b,d)	(d,c)	(c,t)	(d,t)
$c(x,y):$	5	3	3	3	5	2	6	2

Find a flow with value 7 and a cut with capacity 7. What is the value of the maximum flow, and why?

2. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a digraph with source s and sink t and suppose $\phi : \overrightarrow{E} \to \mathbb{R}$ is a function, not necessarily a flow.

(a) Show that

$$\sum_{x \in V} \phi(x, V \setminus x) = \sum_{x \in V} \phi(V \setminus x, x).$$

(b) Deduce from (a) that if ϕ is a flow then the net flow out of s is equal to the net flow into t:

$$\phi(s, V \setminus s) - \phi(V \setminus s, s) = \phi(V \setminus t, t) - \phi(t, V \setminus t).$$

3. Suppose $S \subseteq \overrightarrow{E}$ is a set of edges after whose deletion there is no flow from s to t with strictly positive value. Prove that S contains a cut separating s from t, i.e., there is $X \subset V$ with $s \in X$ and $t \notin X$ such that $\overrightarrow{E}(X, V \setminus X) \subseteq S$.

[Bollobás, *Modern Graph Theory*, III.6 exercise 1]

4. Let $f : \overrightarrow{E} \to \mathbb{R}^+$ be a flow on a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$ with source s, sink t and capacity function $c : \overrightarrow{E} \to \mathbb{R}^+$.

(a) Define the *value* of the flow f.

(b) Suppose that $X \subseteq V$ contains s but not t. Show that the value of f is also equal to

$$f(X, V \setminus X) - f(V \setminus X, X)$$

(c) Using (b) and the fact that $0 \leq f(x,y) \leq c(x,y)$ for each $(x,y) \in \overrightarrow{E}$ prove that the value of f is at most equal to the capacity of a cut $\overrightarrow{E}(X, V \setminus X)$ separating s from t.

[Bollobás, *Modern Graph Theory*, III.6 exercise 2]

5. Let f be a flow on a network comprising digraph $\overrightarrow{G}$, source s, sink t, and capacity function $c : \overrightarrow{E} \to \mathbb{R}^+$.

A *positive cycle* in f is a directed cycle in $\overrightarrow{G}$ such that $f(x, y) > 0$ for each arc (x, y) in the directed cycle.

By successively reducing the number of positive cycles in a given flow f of maximum value prove that there is a maximal flow f^* without positive cycles in which $f^*(V \setminus s, s) = 0$ and $f^*(t, V \setminus t) = 0$.

[Bollobás, *Modern Graph Theory*, III.6 exercise 4]

Supplementary notes Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a digraph and $\overrightarrow{G}$ a *capacity* function $c : \overrightarrow{E} \to \mathbb{R}^+$. The out-neighbourhood of $x \in V$ is

$$\Gamma^+(x) = \{y \in V : (x, y) \in \overrightarrow{E}\},$$

and the in-neighbourhood of x is

$$\Gamma^-(x) = \{z \in V : (z, x) \in \overrightarrow{E}\}.$$

For $X, Y \subseteq V$,

$$\overrightarrow{E}(X, Y) = \{(x, y) \in \overrightarrow{E} : x \in X, y \in Y\}.$$

For a function $\phi : \overrightarrow{E} \to \mathbb{R}^+$ we set

$$\phi(X, Y) = \sum_{\substack{x \in X, y \in Y \\ (x,y) \in \overrightarrow{E}}} \phi(x, y).$$

In particular,

$$\phi(x, V \setminus x) = \sum_{y \in \Gamma^+(x)} \phi(x, y), \quad \text{and} \quad \phi(V \setminus x, x) = \sum_{z \in \Gamma^-(x)} \phi(z, x).$$

The *capacity* of the cut $\overrightarrow{E}(X, V \setminus X)$ is $c(X, V \setminus X)$.

In this notation, a feasible flow is a function $f : \overrightarrow{E} \to \mathbb{R}^+$ such that

$$f(x, y) \le c(x, y) \quad \text{for each} \quad (x, y) \in \overrightarrow{E}, \quad \text{and}$$

$$f(x, V \setminus x) = f(V \setminus x, x) \quad \text{for each} \quad x \in V \setminus \{s, t\}.$$