Combinatorics and Graph Theory I

Exercise sheet 3: Generating functions ctd.

8 March 2017

1. Express the nth term of the sequences given by the following recurrence relations (generalize the method used for the Fibonacci numbers in Section 12.3):

(a) $a_0 = 2,\ a_1 = 3,\ a_{n+2} = 3a_n - 2a_{n+1}\ (n = 0, 1, 2, \dots)$

(b) $a_0 = 0,\ a_1 = 1,\ a_{n+2} = 4a_{n+1} - 4a_n\ (n = 0, 1, 2, \dots)$

(c) $a_0 = 1,\ a_{n+1} = 2a_n + 3\ (n = 0, 1, 2, \dots)$

Since $a_n = 2a_{n-1} + 3$ for $n \geq 1$,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + 3\sum_{n=1}^{\infty} x^n,$$

the g.f. $a(x)$ for (a_n) satisfies

$$a(x) - 1 = 2xa(x) + 3x(1-x)^{-1}$$

from which

$$\begin{aligned} a(x) &= \frac{1}{1-2x} + \frac{3x}{(1-x)(1-2x)} \\ &= \frac{1+2x}{(1-x)(1-2x)} \\ &= \frac{4}{1-2x} - \frac{3}{1-x} \end{aligned}$$

and so

$$a_n = 4 \cdot 2^n - 3.$$

Check: $1 = a_0 = 4 \cdot 1 - 3$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.3.]

2. Solve the recurrence $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$. Find $\lim_{n\to\infty} a_n$.

[*Take base 2 logarithms of the given recurrence.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.5.]

Set $b_n = \log_2 a_n$. Then $b_0 = 1, b_1 = 3$, and, for $n \geq 2$,

$$2b_n = b_{n-1} + b_{n-2}.$$

If $b(x)$ is the g.f for (b_n) then

$$2[b(x) - 1 - 3x] = x[b(x) - 1] + x^2 b(x),$$

from which

$$b(x) = \frac{2+5x}{2-x-x^2} = \frac{2+5x}{(2+x)(1-x)}.$$

Writing as partial fractions,

$$\frac{2+5x}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x}$$

for constants A, B satisfying

$$A + 2B = 2, \quad -A + B = 5,$$

i.e. $A = -\frac{8}{3}, B = \frac{7}{3}$. Then

$$3b(x) = \frac{7}{1-x} - \frac{4}{1+x/2},$$

from which

$$b_n = \frac{7 - 4(-1)^n 2^{-n}}{3}.$$

Check: $1 = b_0 = \frac{7-4}{3}$, $3 = b_1 = \frac{7+4/2}{3}$.

Since $2^{-n} \to 0$ as $n \to \infty$, so $b_n \to \frac{7}{3}$ as $n \to \infty$. By continuity of the function $x \to 2^x$, and $a_n = 2^{b_n}$, we have $\lim a_n = 2^{\lim b_n} = 2^{\frac{7}{3}} = 4\sqrt[3]{2}$.

[The term a_n is the geometric mean of the two previous terms a_{n-1} and a_{n-2} and we have seen that $\lim a_n = 4\sqrt[3]{2}$. Since the geometric mean of two lengths can be constructed using ruler and compass, by constructing first the geometric mean of line segments of lengths 2 and 8 and then iteratively constructing geometric means, in the limit you reach a construction of the cube root of 2. A famous theorem of Euclidean geometry is that the cube root of 2 cannot be constructed by ruler and compass alone in a finite number of steps.]

3.

(a) Solve the recurrence $a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$ with the initial condition $a_0 = 1$.

For $n \geq 2$,

$$a_{n-1} = a_{n-2} + \cdots + a_1 + a_0$$

so that

$$a_n = a_{n-1} + a_{n-1} = 2a_{n-1},$$

whence $a_n = 2^{n-1}$ for $n \geq 1$, by induction on n, or by finding the generating function $a(x)$ satisfies $a(x) - 1 - x = 2x(a(x) - 1)$, from which $a(x) = \frac{1-x}{1-2x}$ and then reading off from this that $a_n = 2^n - 2^{n-1} = 2^{n-1}$ for $n \geq 1$.

Alternatively, using the fact that if $a(x)$ is the g.f. for (a_n) then $\frac{a(x)}{1-x}$ is the g.f. for the partial sums $(\sum_{i=0}^{n} a_i)$, and so $\frac{xa(x)}{1-x}$ is the g.f. for the partial sums $(\sum_{i=0}^{n-1} a_i)$, we have

$$a(x) - 1 = \frac{xa(x)}{1-x},$$

from which $a(x) = \frac{1-x}{1-2x}$.

*(b) Solve the recurrence $a_n = a_{n-1} + \ a_{n-3} + \cdots + a_1 + a_0$ $(n \geq 3)$ with the initial condition $a_0 = a_1 = a_2 = 1$.

Beginning with a_0, the first few terms of this sequence are $1, 1, 1, 2, 4, 7, 12, 21, 37, 65, \ldots$. For some examples of what a_n counts see `https://oeis.org/A005251` (e.g. the number of compositions of n avoiding the part 2; so $a_4 = 4$ since the compositions of 4 avoiding 2 are 4, $3+1$, $1+3$ and $1+1+1+1$).

Since for $n \geq 3$

$$a_n = \left(\sum_{i=0}^{n-1} a_i \right) - a_{n-2},$$

the g.f for (a_n) satisfies

$$a(x) - 1 - x - x^2 = \frac{x[a(x) - 1 - x]}{1 - x} - x^2[a(x) - 1],$$

from which, after a little algebra,

$$a(x)[1 - 2x + x^2 - x^3] = 1 - x,$$

and the g.f. for (a_n) is thus

$$a(x) = \frac{1 - x}{1 - 2x + x^2 - x^3}.$$

[Alternative derivation: for $n \geq 3$, $a_n = a_{n-1} + a_{n-3} + (a_{n-1} - a_{n-2}) = 2a_{n-1} - a_{n-2} + a_{n-3}$, from which $a(x) - 1 - x - x^2 = 2x[a(x) - 1 - x] - x^2[a(x) - 1] + x^3 a(x)$, and this yields the same formula for $a(x)$.]

The denominator does not factorize easily as polynomial in x, but rather in $x^{\frac{1}{2}}$ as difference of two squares:

$$1 - 2x + x^2 - x^3 = (1 - x)^2 - (x^{\frac{3}{2}})^2 = [1 - x - x^{\frac{3}{2}}][1 - x + x^{\frac{3}{2}}].$$

We find then that

$$2a(x) = \frac{1}{1 - x - x^{\frac{3}{2}}} + \frac{1}{1 - x + x^{\frac{3}{2}}}.$$

Expanding the two series of the form $(1 - y)^{-1}$ with $y = x(1 \pm x^{\frac{1}{2}})$,

$$\begin{aligned} 2a(x) &= \sum_{j=0}^{\infty} x^j [(1 + x^{\frac{1}{2}})^j + (1 - x^{\frac{1}{2}})^j] \\ &= 2 \sum_{j=0}^{\infty} x^j \sum_{i=0}^{j} \binom{j}{2i} x^i \\ &= 2 \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n-i}{2i} \right) x^n, \end{aligned}$$

from which

$$a_n = \sum_{i=0}^{n} \binom{n-i}{2i},$$

in which the range of summation can in fact be restricted to $0 \leq i \leq \lfloor n/3 \rfloor$ since the binomial coefficient is zero when $2i > n - i$. As a check,

$$a_0 = \binom{0}{0} = 1, \quad , a_1 = \binom{1}{0} = 1, \quad a_2 = \binom{2}{0} = 1, \quad a_3 = \binom{3}{0} + \binom{2}{2} = 2.$$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.6.]

4. Express the sum

$$S_n = \binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n}$$

as the coefficient of x^{2n} in a suitable power series. Find a simple formula for S_n.

[*Use* $x^k(1+2x)^k = \sum_{i=0}^{k} 2^i\binom{k}{i}x^{k+i}$, *sum over integers* k, *and set* $k+i=2n$ *to pick out the requisite coefficient.*] [Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercise 12.3.7]

$$\frac{1}{1-x-2x^2} = \sum_{k=0}^{\infty} x^k(1+2x)^k = \sum_{k=0}^{\infty}\sum_{i=0}^{k} 2^i\binom{k}{i}x^{k+i}$$
$$= \sum_{m=0}^{\infty}\sum_{i=0}^{m} 2^i\binom{m-i}{i}x^m,$$

in which the range in the inner summation can in fact be restricted to $0 \le i \le \lfloor m/2 \rfloor$ since $\binom{m-i}{i} = 0$ for $i > m-i$. Taking $m = 2n$,

$$[x^{2n}]\,\frac{1}{1-x-2x^2} = \sum_{i=0}^{n} 2^i\binom{2n-i}{i} = S_n.$$

Since

$$\frac{1}{1-x-2x^2} = \frac{1}{(1+x)(1-2x)} = \frac{1}{3}\left(\frac{1}{1+x} + \frac{2}{1-2x}\right),$$

we have

$$S_n = \frac{1}{3}(-1)^{2n} + \frac{2}{3}2^{2n} = \frac{1}{3}(1+2^{2n+1}).$$

Check: $\binom{0}{0} = 1 = S_0 = \frac{1}{3}(1+2)$ and $\binom{2}{0} + 2\binom{1}{1} = 3 = S_1 = \frac{1}{3}(1+8)$.

5.

(a) Show that the number $\frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n$ is an integer for all $n \ge 1$. [*Find the generating function for the sequence; equivalently, the recurrence it satisfies.*]

Let $a(x)$ be the g.f. for the sequence (a_n) defined by $a_n = \frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n$.

$$a(x) = \sum_{n=0}^{\infty}\left[\frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n\right]x^n = \frac{\frac{1}{2}}{1-(1+\sqrt{2})x} + \frac{\frac{1}{2}}{1-(1-\sqrt{2})x}$$
$$= \frac{1-x}{1-2x-x^2}$$

The initial terms are $a_0 = 1, a_1 = 1$. Since

$$a(x) = 1 - x + 2xa(x) + x^2a(x),$$

we have

$$a(x) - 1 - x = 2x[a(x) - 1] + x^2a(x),$$

from which, for $n \ge 2$,

$$a_n = 2a_{n-1} + a_{n-2}.$$

Remark: Here we use the fact that $a(x) - a_0 - a_1x$ is the g.f. for $(a_n)_{n\ge2}$, $x[a(x) - a_0]$ is the g.f. for $(a_{n-1})_{n\ge2}$ and $x^2a(x)$ is the g.f. for $(a_{n-2})_{n\ge2}$.

Since $a_0 = 1 = a_1$ are integers and the recurrence $a_n = 2a_{n-1} + a_{n-2}$ is an integer linear combination, it follows that a_n is an integer for all n.

Alternatively, a direct appeal to the binomial theorem gives

$$\begin{aligned}\frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n &= \frac{1}{2}\left(\sum_{i=0}^{n}\binom{n}{i}2^{i/2}(1+(-1)^i)\right)\\ &= \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^j,\end{aligned}$$

which is a sum of postive integers.

(b) Show that the decimal expansion of $(6+\sqrt{37})^{999}$ has at least 999 zeros following the decimal point.

[*Show that* $a_n = (6+\sqrt{37})^n + (6-\sqrt{37})^n$ *satisfies* $a_{n+2} = 12a_{n+1} + a_n$ *with initial conditions* $a_0 = 2$, $a_1 = 12$, *and hence is an integer for all* $n \geq 1$. *Use the fact that* $\sqrt{37} - 6 < 0 \cdot 1$.]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.3, exercises 12.3.9 and 12.3.10.]

Let $a_n = (6+\sqrt{37})^n + (6-\sqrt{37})^n$ have g.f. $a(x)$. Then

$$\begin{aligned}a(x) = \sum_{n=0}^{\infty}\left[(6+\sqrt{37})^n + (6-\sqrt{37})^n\right]x^n &= \frac{1}{1-(6+\sqrt{37})x} + \frac{1}{1-(6-\sqrt{37})x}\\ &= \frac{2-12x}{1-12x-x^2}\end{aligned}$$

Calculating $a_0 = 2, a_1 = 12$, and

$$a(x) = 2 - 12x + 12xa(x) + x^2a(x),$$

$$a(x) - 2 - 12x = 12x[a(x) - 2] + x^2a(x),$$

we deduce that, for $n \geq 2$,

$$a_n = 12a_{n-1} + a_{n-2}.$$

It follows that (a_n) is an integer sequence.

Given that $0 < \sqrt{37} - 6 = \frac{1}{\sqrt{37}+6} < \frac{1}{12} < \frac{1}{10}$, we have $(6+\sqrt{37})^n = a_n - (6-\sqrt{37})^n$. When n is odd,

$$0 < -(6-\sqrt{37})^n = (\sqrt{37}-6)^n < 10^{-n},$$

so that $(6+\sqrt{37})^n$ is equal to the integer a_n plus a positive number with at least n zeroes after the decimal point.