Combinatorics and Graph Theory I

Exercise sheet 2: Generating functions

1 March 2017

2. In this question we use the convolution formula for coefficients when multiplying polynomials (which extends to power series):

$$\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k,$$

in which we set $a_i = 0$ for $i \notin \{0,1,\dots,n\}$ and $b_j = 0$ for $j \notin \{0,1,\dots,m\}$. (This latter stipulation is not necessary for power series - and indeed, if we consider polynomials simply as power series with finitely many non-zero coefficients, we do not even have to mention this.)

(a) Explain why $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$.

This is clear from the formula $\binom{n}{i} = \frac{n!}{i!(n-i)!} = \binom{n}{n-i}$. By the interpretation of $\binom{n}{i}$ for postive integer n as the number of subsets of $[n]$ of size i:

$$\binom{n}{i} = |\{S \subseteq [n] : |S| = i\} = |\{S \subseteq [n] : |[n]\backslash S| = n-i\} = |\{T \subseteq [n] : |T| = n-i\} = \binom{n}{n-i},$$

using the bijection $S \leftrightarrow [n] \setminus S = T$ between $\binom{[n]}{i}$ and $\binom{[n]}{n-i}$.

Find a closed formula (i.e., not involving summation) for

$$\sum_{i=0}^{n} \binom{n}{i}^2.$$

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i}^2 &= \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} \\ &= [x^n]\left(\sum_{i=0}^{n}\binom{n}{i}x^i\right)^2 \\ &= [x]^n(1+x)^{2n} \\ &= \binom{2n}{n}.\end{aligned}$$

(b) Find a closed formula for

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}^2.$$

[*The answer depends on the parity of n.*]

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\binom{n}{i}^2 &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n}{n-i} \\
&= [x^n]\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}x^i\right)\left(\sum_{i=0}^{n}\binom{n}{i}x^i\right) \\
&= [x]^n(1+x)^n(1-x)^n \\
&= [x]^n(1-x^2)^n \\
&= \begin{cases}(-1)^{n/2}\binom{n}{n/2} & n \text{ even,}\\ 0 & n \text{ odd.}\end{cases}
\end{aligned}$$

(c) By determining the coefficient of x^k in the expansion of $(1+x)^m(1+x)^{n-m} = (1+x)^n$ show that

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n-m}{k-i} = \binom{n}{k}.$$

By the Binomial Theorem and the convolution formula for multiplying polynomials, for $0 \le k \le m+n-m = n$,

$$\begin{aligned}
[x]^k\ (1+x)^m(1+x)^{n-m} &= [x]^k\ \left(\sum_{i=0}^{m}\binom{m}{i}x^i\right)\left(\sum_{i=0}^{n-m}\binom{n-m}{i}x^i\right) \\
&= \sum_{i=0}^{k}\binom{m}{i}\binom{n-m}{k-i},
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[x]^k\ (1+x)^m(1+x)^{n-m} &= [x]^k\ (1+x)^n \\
&= [x]^k\ \left(\sum_{i=0}^{n}\binom{n}{i}x^i\right) \\
&= \binom{n}{k}.
\end{aligned}$$

(d) What identity results upon setting $m=1$ in the identity of part (c)?

For $m=1$ the identity states that

$$\binom{1}{0}\binom{n-1}{k} + \binom{1}{1}\binom{n-1}{k-1} = \binom{n}{k},$$

which is Pascal's recurrence formula.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.1, another proof of Prop. 3.3.4, and exercise 12.1.6 extended.]

3. For this question we use the binomial expansion

$$(1+x)^a = 1 + \sum_{k=1}^{\infty}\frac{a(a-1)\cdots(a-k+1)}{k!}x^k,$$

valid for all $a \in \mathbb{R}$ (and $|x| < 1$ for convergence to a function of $x \in \mathbb{R}$). Extending the definition of the binomial coefficient to real numbers by setting, for $k \geq 1$, $\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}$, and $\binom{a}{0} = 1$, the binomial expansion is more compactly written as

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k.$$

In particular, for a positive integer m,

$$\begin{aligned}(1-x)^{-m} &= 1 + \sum_{k=1}^{\infty} \frac{(-m)(-m-1)\cdots(-m-k+1)}{k!}(-x)^k \\ &= 1 + \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k \\ &= \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} x^k\end{aligned}$$

For example, taking $m = 2$,

$$(1-x)^{-2} = 1 + \sum_{k=1}^{\infty} \binom{k+1}{1} x^k = 1 + 2x + 3x^2 + \cdots$$

For a polynomial or power series $a(x)$ we let $[x^k]a(x)$ denote the ceofficient of x^k in $a(x)$. Determine the following coefficients:

(a) $[x^{50}]\,(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$

We have

$$x^7 + x^8 + x^9 + x^{10} + \cdots = x^7(1 + x + x^2 + \cdots) = x^7(1-x)^{-1}$$

so that

$$(x^7 + x^8 + x^9 + x^{10} + \cdots)^6 = x^{42}(1-x)^{-6}$$

and by the Binomial Theorem

$$\begin{aligned}(1-x)^{-6} &= 1 + \frac{(-6)}{1}(-x) + \frac{(-6)(-7)}{2!}(-x)^2 + \frac{(-6)(-7)(-8)}{3!}(-x)^3 + \cdots \\ &= 1 + 6x + 21x^2 + 56x^3 + \cdots\end{aligned}$$

in which the coefficient of x^8 is

$$(-1)^8 \frac{(-6)(-7)(-8)(-9)(-10)(-11)(-12)(-13)}{8!} = 1287.$$

Hence $[x^{50}]\,(x^7 + x^8 + x^9 + x^{10} + \cdots)^6 = 1287$.

(b) $[x^5](1-2x)^{-2}$

By the Binomial Theorem, for $|x| < \frac{1}{2}$,

$$(1-2x)^{-2} = 1 + \sum_{k=1}^{\infty} \binom{k+1}{1} (2x)^k,$$

in which the coefficient of x^5 is $6 \cdot 2^5 = 192$.

(c) $[x^4]\,\sqrt[3]{(1+x)}$

(d) $[x^3]\,(2+x)^{3/2}/(1-x)$

(e) $[x^3](1-x+2x^2)^9$ Method 1: Using the multinomial expansion

$$(1-x+2x^2)^9 = \sum_{i+j+k=9}^{9} \binom{9}{i,j,k}(-x)^j(2x^2)^k$$

and to obtain exponent 3 we require $j+2k=3$ (only possiblities $j=1, k=1$ and $j=3, k=0$), so

$$\begin{aligned}
[x^3]\,(1-x+2x^2)^9 &= \sum_{\substack{i+j+k=9\\ j+2k=3}} \binom{9}{i,j,k}(-1)^j 2^k \\
&= \binom{9}{7,1,1}(-1)2^1 + \binom{9}{6,3,0}(-1)^3 2^0 \\
&= -2\binom{9}{7,1,1} - \binom{9}{6} = -4\binom{9}{2} - \binom{9}{3} = -228.
\end{aligned}$$

Method 2: By the binomial expansion,

$$\begin{aligned}
(1-x+2x^2)^9 &= \sum_{i=0}^{9}\binom{9}{i}(-x+2x^2)^i \\
&= 1 + 9(-x+2x^2) + \binom{9}{2}(-x+2x^2)^2 + \binom{9}{3}(-x+2x^2)^3 + \cdots \\
&= 1 + 9(-x+2x^2) + \binom{9}{2}(x^2-4x^3+4x^4) + \\
&\quad \binom{9}{3}[-x^3+3(-x)^2(2x^2)+3(-x)(2x^2)^2+(2x^2)^3] + \cdots
\end{aligned}$$

where powers of x in the ellipses (...) are higher than x^3. Hence the coefficient of x^3 is

$$-4\binom{9}{2} - \binom{9}{3} = -228.$$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.2]

5.

(a) Let $A, B, C \subset \mathbb{N}$ and let $a(x) = \sum_{i\in A} x^i$, $b(x) = \sum_{j\in B} x^j$ and $c(x) = \sum_{k\in C} x^k$ be power series. Explain why the number of solutions to the equation $i+j+k=n$ with $i \in A$, $j \in B$ and $k \in C$ is equal to the coefficient of x^n in the power series $a(x)b(x)c(x)$.

(b) Let a_n be the number of solutions to the equation

$$i+3j+3k=n, \qquad i \geq 0, j \geq 1, k \geq 1.$$

Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and derive a formula for a_n.

The g.f. is

$$(1+x+x^2+\cdots)(x^3+x^6+x^9+\cdots)^2 = (1-x)^{-1}x^6(1-x^3)^{-2}.$$

Using $1 - x^3 = (1-x)(1+x+x^2)$, this is

$$x^6(1-x)^{-3}(1+x+x^2)^{-2} = x^6(1+x+x^2)(1-x^3)^{-3}.$$

(The latter form is preferred as it involves just one infinite power series expansion.) By the Binomial Theorem,

$$(1-x^3)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^{3k} = 1 + 3x^3 + 6x^6 + 10x^9 + \cdots$$

Then

$$\begin{aligned} x^6(1+x+x^2)(1-x^3)^{-3} &= \sum_{k=0}^{\infty} \binom{k+2}{2} (x^{3k+6} + x^{3k+7} + x^{3k+8}) \\ &= \sum_{j=2}^{\infty} \binom{j}{2} (x^{3j} + x^{3j+1} + x^{3j+2}) \\ &= \sum_{i=6}^{\infty} \binom{\lfloor i/3 \rfloor}{2} x^i. \end{aligned}$$

Hence $a_n = \binom{\lfloor n/3 \rfloor}{2}$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.5]