Combinatorics and Graph Theory I

Exercise sheet 2: Generating functions

1 March 2017

1.

(a) Prove the multinomial theorem, for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,k_2,\dots,k_m \in \mathbb{N}}} \binom{n}{k_1, k_2, \dots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m},$$

by a combinatorial argument similar to the proof of the binomial expansion given in Section 12.1. Here

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is the *multinomial coefficient.*

[*Compare Theorem 3.3.5 and its proof by induction in Exercise 3.3.26.*]

(b) Let a_n be the number of ordered m-tuples $(k_1, k_2, \dots, k_m)$ of nonnegative integers with $k_1 + k_2 + \cdots + k_m = n$. [*Thus a_n is the number of terms on the right-hand side of the multinomial expansion in (a).*] Find the generating function of the sequence $(a_0, a_1, a_2, \dots)$.

(c) Use (b) and the generalized binomial theorem to derive a formula for a_n.

[*Compare also the combinatorial proof of (c) given in Section 3.3.*]

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.1, exercise 12.1.3 and section 12.2, exercise 12.2.6]

2.

(a) Explain why $\binom{n}{i}^2 = \binom{n}{i}\binom{n}{n-i}$. Find a closed formula (i.e., not involving summation) for

$$\sum_{i=0}^{n} \binom{n}{i}^2$$

(b) Find a closed formula for

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}^2.$$

[*The answer depends on the parity of n.*]

(c) By determining the coefficient of x^k in the expansion of $(1+x)^m(1+x)^{n-m} = (1+x)^n$ show that

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n-m}{k-i} = \binom{n}{k}.$$

(d) What identity results upon setting $m = 1$ in the identity of part (c)?

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.1, another proof of Prop. 3.3.4, and exercise 12.1.6 extended.]

3. For a polynomial or power series $a(x)$ we let $[x^k]a(x)$ denote the ceofficient of x^k in $a(x)$. Determine the following coefficients:

(a) $[x^{50}]\,(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$

(b) $[x^5](1 - 2x)^{-2}$

(c) $[x^4]\,\sqrt[3]{(1 + x)}$

d) $[x^3]\,(2 + x)^{3/2}/(1 - x)$

(e) $[x^3](1 - x + 2x^2)^9$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.2]

4. Find generating functions for the following sequences (express them in closed form, without infinite series!):

(a) $0, 0, 0, 0, -6, 6, -6, 6, -6, \ldots$

(b) $1, 0, 1, 0, 1, 0, \ldots$

(c) $1, 2, 1, 4, 1, 8, \ldots$

(d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.3]

5.

(a) Let $A, B, C \subset \mathbb{N}$ and let $a(x) = \sum_{i \in A} x^i$, $b(x) = \sum_{j \in B} x^j$ and $c(x) = \sum_{k \in C} x^k$ be power series. Explain why the number of solutions to the equation $i + j + k = n$ with $i \in A$, $j \in B$ and $k \in C$ is equal to the coefficient of x^n in the power series $a(x)b(x)c(x)$.

(b) Let a_n be the number of solutions to the equation

$$i + 3j + 3k = n, \qquad i \geq 0, j \geq 1, k \geq 1.$$

Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and derive a formula for a_n.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, exercise 12.2.5]

6.

(a) Find the generating function for the sequence $(a_0, a_1, a_2, \ldots)$ with $a_n = (n + 1)^2$.

(b) Check that if $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \ldots)$ then $\frac{1}{1-x}a(x)$ is the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$.

(c) Using (a) and (b) calculate the sum $\sum_{k=1}^{n} k^2$.

(d) By a similar method, calculate the sum $\sum_{k=1}^{n} k^3$.

(e) For natural numbers n and m, find a closed formula for the sum $\sum_{k=0}^{m} (-1)^k \binom{n}{k}$.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 12.2, Problem 2.2.4, exercise 12.2.9.]