Combinatorics and Graph Theory I

Exercise sheet 1: Estimates – solutions to selected exercises

22 February 2017

1. Show that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ then $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$ and $f_1(n)f_2(n) = O(g_1(n)g_2(n))$.

Express in words the statements $f(n) = O(1)$, $g(n) = \Omega(1)$ and $h(n) = n^{O(1)}$.

$f(n) = O(1)$ says that f is bounded (by a constant): $|f(n)| \leq C$ for some constant C.

$g(n) = \Omega(1)$ says that g is bounded away from 0: $|g(n)| \geq C$ for some constant $C > 0$.

$h(n) = n^{O(1)}$ says that h grows no faster than a polynomial: $|h(n)| \leq n^C$ for some constant C (so $h = O(n^d)$ for $d = \lceil C \rceil$, i.e., no faster than a polynomial of degree d, for some d).

(a) Prove that $n^\alpha = O(n^\beta)$ for $\alpha \leq \beta$.

(b) Prove that $n^\gamma = O(a^n)$ for any $a > 1$.

We show that the sequence $(\frac{n^\gamma}{a^n})$ is decreasing for sufficiently large n, say $n \geq m$, from which it follows that $\frac{n^\gamma}{a^n} \leq \frac{m^\gamma}{a^m}$, i.e. $n^\gamma \leq \frac{m^\gamma}{a^m} \cdot a^n$ for $n \geq m$. To see this, consider the quotient of successive terms,

$$\frac{(n+1)^\gamma / a^{n+1}}{n^\gamma / a^n} = \left(\frac{n+1}{n}\right)^\gamma \frac{1}{a},$$

which, since $\frac{1}{a} < 1$ and $\left(\frac{n+1}{n}\right)^\gamma \to 1$ as $n \to \infty$, is less than 1 for $n \geq m$ for suitable choice of m. From this point on the sequence $(\frac{n^\gamma}{a^n})$ decreases.

An alternative way to show $n^\gamma = O(a^n)$ is to show the stronger statement that $\frac{n^\gamma}{a^n} \to 0$ as $n \to \infty$, i.e. $n^\gamma = o(a^n)$, from which it follows that $n^\gamma < a^n$ for sufficiently large n. (There are many ways to prove that $\frac{n^\gamma}{a^n} \to 0$, for example using L'Hôpital's rule from calculus to show that $\frac{x^\gamma}{a^x} \to 0$ as $x \to \infty$ for real variable x; we choose a proof avoiding the need to move into calculus of a real variable.)

We may assume $\gamma \geq 1$ (since $n^\gamma \leq n$ otherwise and $\frac{n^\gamma}{a^n} \leq \frac{n}{a^n}$ so once we prove the result for $\gamma = 1$ then we are done). Taking the γth root of $\frac{n^\gamma}{a^n}$,

$$\frac{n}{a^{n/\gamma}} = \frac{n}{b^n},$$

where $b = a^{1/\gamma} > 1$. Thus it suffices to prove that $\frac{n}{b^n} \to 0$ for $b > 1$. Let $b = 1 + c$ (in which $c > 0$). By the Binomial Theorem, for $n \geq 2$,

$$b^n = (1+c)^n \geq 1 + cn + n(n-1)c^2/2 > c^2 n(n-1)/2.$$

Hence

$$\frac{n}{b^n} < \frac{n}{c^2 n(n-1)/2} = \frac{2}{c^2(n-1)} \to 0 \quad \text{as } n \to \infty.$$

(c) Deduce from (b) that $(\ln n)^\gamma = O(n^\alpha)$ for any $\alpha > 0$.

Lemma. Let $f, g : (0,\infty) \to \mathbb{R}$ and let $h : (0,\infty) \to \mathbb{R}$ be monotone increasing. If $f(x) = O(g(x))$ then $f(h(x)) = O(g(h(x)))$.

Proof. If $|f(x)| \le Cg(x)$ for $x \ge x_0$ and h is increasing then $h(x) \ge h(x_0) \ge x_0$ for $x \ge x_0$ so that $|f(h(x))| \le Cg(h(x))$ for $x \ge x_0$.

The function $\ln : (0,\infty) \to \mathbb{R}$ is monotone increasing. By the previous lemma and (b) extended from n to a real variable x (i.e. $x^\gamma = O(a^x)$ for any $a > 1$) we have $(\ln n)^\gamma = O(a^{\ln n})$ for any $a > 1$. With

$$a^{\ln n} = (e^{\ln a})^{\ln n} = e^{\ln a \ln n} = (e^{\ln n})^{\ln a} = n^{\ln a},$$

set $\alpha = \ln a$, which can be made arbitrarily close to 0 by taking a arbitrarily close to 1.

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.4, Fact 3.4.3 and exercise 3.4.6.]

5.

(a) Prove the arithmetic-geometric mean inequality $\sqrt{ab} \le \frac{1}{2}(a+b)$.

(b) Prove by induction on n and using (a) that for $n \ge 1$ we have

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \le 2\sqrt{n} - 1.$$

[*This can alternatively be obtained by integration as in question 3(b).*]

Base case $n = 1$:

$$2\sqrt{2} - 2 = 2(\sqrt{2} - 1) < 2(\frac{1}{2}) = 1 = \frac{1}{\sqrt{1}} \le 2\sqrt{1} - 1 = 1,$$

where we use $\sqrt{2} - 1 < \frac{1}{2}$, which follows from $2 < (\frac{3}{2})^2 = \frac{9}{4}$.

Assume true for given $n \ge 1$, i.e. that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n+1} + 2 > 0 \tag{1}$$

and

$$2\sqrt{n} - 1 - \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \ge 0. \tag{2}$$

Then the left-hand side of inequality (1) with $n+1$ in place of n is equal to

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} - 2\sqrt{n+2} + 2 &= \frac{1}{\sqrt{n+1}} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n+2} + 2 \\
&> \frac{1}{\sqrt{n+1}} + 2\sqrt{n+1} - 2 \ - 2\sqrt{n+2} + 2 \\
&= \frac{1}{\sqrt{n+1}} + 2(\sqrt{n+1} - \sqrt{n+2}) \\
&= \frac{1 + 2(n+1) - 2\sqrt{(n+1)(n+2)}}{\sqrt{n+1}} \\
&\ge 0,
\end{aligned}$$

where we use the induction hypothesis for the first inequality and for the last inequality we apply (a) to find that

$$2\sqrt{(n+1)(n+2)} \leq 2n+3.$$

This establishes the induction step for inequality (1).

Similarly, the left-hand side of inequality (2) with $n+1$ in place of n is equal to

$$\begin{aligned}
2\sqrt{n+1} - 1 - \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} &= 2\sqrt{n+1} - 1 - \frac{1}{\sqrt{n+1}} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \\
&\geq 2\sqrt{n+1} - 1 - \frac{1}{\sqrt{n+1}} - 2\sqrt{n} + 1 \\
&= 2(\sqrt{n+1} - \sqrt{n}) - \frac{1}{\sqrt{n+1}} \\
&= \frac{2(n+1) - 2\sqrt{n(n+1)} - 1}{\sqrt{n+1}} = \frac{2n+1-2\sqrt{n(n+1)}}{\sqrt{n+1}} \\
&\geq 0,
\end{aligned}$$

where we use the induction hypothesis for the first inequality and for the last inequality we apply (a) to find that

$$2\sqrt{n(n+1)} \leq 2n+1.$$

This establishes the induction step for inequality (2).

(c) Use the inequality $1+x \leq e^x$ and induction to prove the inequality in 3(a). The inequality in 3(a) is that, for $n \geq 1$,

$$\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1.$$

Thus we wish to prove

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) > 0 \tag{3}$$

and

$$\ln(n) + 1 - \sum_{k=1}^{n} \frac{1}{k} \geq 0. \tag{4}$$

For $n = 1$ we have $\ln 2 < 1 \leq 1$.

Assume as inductive hypothesis the inequalities (3) and (4) for given $n \geq 1$; we wish to show they hold for $n+1$ as well.

The left-hand side of (3) with $n+1$ in place of n is

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k} - \ln(n+2) &= \frac{1}{n+1} + \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) + \ln(n+1) - \ln(n+2) \\
&> \frac{1}{n+1} + \ln(n+1) - \ln(n+2) = \frac{1}{n+1} - \ln\frac{n+2}{n+1} \\
&= \frac{1}{n+1} - \ln(1 + \frac{1}{n+1}) \\
&\geq 0,
\end{aligned}$$

in which the first inequality is by the induction hypothesis and the second is by the inequality $1+x \leq e^x$ with $x = \frac{1}{n+1}$, which by taking logarithms gives $\ln(1+\frac{1}{n+1}) \leq \frac{1}{n+1}$.

The left-hand side of (4) with $n+1$ in place of n is

$$\begin{aligned}
\ln(n+1)+1-\sum_{k=1}^{n+1}\frac{1}{k} &= \ln(n+1)+\ln(n)+1-\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{n+1}-\ln(n) \\
&\geq \ln(n+1)-\frac{1}{n+1}-\ln(n) \\
&= \ln(1+\frac{1}{n})-\frac{1}{n+1}
\end{aligned}$$

and to complete the proof of the induction step for equation (4) we require the inequality

$$\ln(1+\frac{1}{n}) \geq \frac{1}{n+1}.$$

To show this we prove the equivalent inequality $1+\frac{1}{n} \geq e^{\frac{1}{n+1}}$. This follows from

$$\begin{aligned}
1+\frac{1}{n} = \frac{n+1}{n} &= \frac{1}{1-\frac{1}{n+1}} \\
&\geq \frac{1}{e^{-\frac{1}{n+1}}} \\
&= e^{\frac{1}{n+1}}.
\end{aligned}$$

[Matoušek & Nešetřil, *Invitation to Discrete Mathematics*, section 3.5, exercises 3.5.12 and 3.5.13.]